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Linear response and hydrodynamics for granular fluids

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A formal derivation of linear hydrodynamics for a granular fluid is given. The linear response to small spatial perturbations of a homogeneous reference state is studied in detail, using methods of nonequilibrium statistical mechanics. A transport matrix for macroscopic excitations in the fluid is defined in terms of the response functions. An expansion in the wave vector to second order allows identification of all phenomenological susceptibilities and transport coefficients through Navier-Stokes order in terms of appropriate time correlation functions. The transport coefficients in this representation are the generalization to granular fluids of the familiar Helfand and Green-Kubo relations for normal fluids. The analysis applies to a variety of collision rules. Important differences in both the analysis and results from those for normal fluids are identified and discussed. A scaling limit is described corresponding to the conditions under which idealized inelastic hard sphere models can apply. Further details and interpretation are provided in the paper following this one, by specialization to the case of smooth, inelastic hard spheres with constant coefficient of restitution.

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I. INTRODUCTION

Forty years ago, significant advances in the theory and simulation of simple atomic fluids were stimulated by the application of formally exact methods from nonequilibrium statistical mechanics, namely, linear response and the "time correlation function method" [1]. The results differ from those of earlier studies based on approximate kinetic theories in that they are formally exact and closely related to properties measured in experiments. Subsequently, a great deal has been learned through the study of appropriate time correlation functions by theory, simulation, and experiment [2]. In many respects, the more recent study of granular fluids is poised to exploit this body of work on normal fluids. Significant advances have been made in the past decade through the application of molecular dynamics simulations and kinetic theory. However, although the generalization of the formal structure for nonequilibrium statistical mechanics has been described [3,4], relatively few applications outside simulation and kinetic theory [5–7] have been given. In particular, relations between properties of interest and appropriate time correlation functions appear restricted to the simplest cases of tagged particle motion [8–11] and liquid structure [12].

The objective of this work is to provide a first step toward the general application of these formal methods to granular fluids. The response of an isolated granular fluid to small perturbations of a reference homogeneous state is studied and used to extract formally exact expressions for the hydrodynamic transport coefficients up to Navier-Stokes order. It is the analog of the study of excitations about the equilibrium Gibbs state for normal fluids. In both cases, it is achieved by formulating the problem of linear response at two levels, first using phenomenological hydrodynamics and then from statistical mechanics. The response of the hydrodynamic fields $y_{\alpha}(\boldsymbol{r},t)$ (local number density, granular temperature, and flow velocity) to small initial spatial deviations $\delta y_{\alpha}(\boldsymbol{r},0)$ from their values in a homogeneous (formally infinite) reference state is written in terms of a matrix of response functions $C_{\alpha\beta}(\boldsymbol{r};t)$

$$\delta y_{\alpha}(\boldsymbol{r},t)=\int d\boldsymbol{r}'\,C_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}';t)\,\delta y_{\beta}(\boldsymbol{r}',0). \tag{1}$$

Here and below a summation convention is used for repeated pairs of variables, unless otherwise indicated. At the level of the phenomenological hydrodynamics, the $C_{\alpha\beta}(\boldsymbol{r};t)$ are parametrized by the pressure, cooling rate, and the various transport coefficients such as the viscosity and thermal conductivity. At the level of statistical mechanics, the $C_{\alpha\beta}(\boldsymbol{r};t)$ are time correlation functions for the homogeneous reference state about which the response is measured. On the length and time scales at which hydrodynamics is expected to be valid, the two descriptions should coincide, allowing an identification of the various hydrodynamic parameters in terms of corresponding correlation functions over the homogeneous reference state. This first formal step gives the precise link between the macroscopic quantities of interest and the underlying microscopic laws governing the fluid and the reference state considered. Subsequent detailed many-body analysis is required for explicit evaluation of these expressions in terms of the state macroscopic variables (e.g., density and temperature). Only the first step is completed here. A kinetic theory representation of these results is described and applied elsewhere [13]. The utility of such formal results for both simulation and theory also has been illustrated recently for granular fluids in the cases of mobility [10] and impurity diffusion [11].

There are two closely related logical components to this prescription. The first is a demonstration that the statistical mechanical representation admits a limit with the same form as that from hydrodynamics, allowing the above identification of the hydrodynamic parameters. The second is a proof that the hydrodynamic description dominates all other possible excitations in this limit. The first demonstration constitutes the usual derivation of Helfand [14] and Green-Kubo [15] time correlation function expressions for transport coefficients in a normal fluid, and the presentation here is essentially its extension to granular fluids. The precise characterization of conditions for its dominance is more difficult and remains incomplete even for normal fluids. The qualitative argument for normal fluids is that hydrodynamics persists on the longest time scale since it describes the dynamics of the densities of local conserved quantities, with relaxation times that scale with the wavelength of the perturbation. Hence as the system approaches homogeneity all other excitations decay to zero, leaving a space and time scale on which only hydrodynamics remains. A similar assumption is made here for granular fluids, although there is a new time scale associated with the rate of energy loss which does not scale with the wavelength. Nevertheless it is possible that this faster scale remains slow compared to the microscopic excitations. This makes a study of the context for hydrodynamics even more significant in the case of granular fluids.

Although the above prescription for application of linear response to the hydrodynamics of a granular fluid is simple to state in general, its implementation in detail requires addressing a number of differences from the case of normal fluids. In granular fluids, there is no "approach to equilibrium" in the usual sense because there is no stationary equilibrium Gibbs state, due to the continual loss of energy by inelastic collisions. Thus Eq. (1) constitutes a reformulation of Onsager's observation that linear nonequilibrium regression laws can be studied via equilibrium fluctuations [16,17]. Instead, the regression laws are those relative to a different "universal" homogeneous state for an isolated granular fluid, referred to as the homogeneous cooling state (HCS). In contrast to the Gibbs state, it is not simply a function of the global invariants nor is it stationary. However, all of the homogeneous state time dependence occurs through the average energy (or, equivalently, the granular temperature). It is universal in the sense that a wide class of homogeneous states are observed in simulations to approach the HCS after a few collisions within the statistical uncertainties. The objective here is to formulate the hydrodynamic response to small spatial perturbations of this special homogeneous but nonequilibrium, nonstationary HCS.

For linear response, it is sufficient to consider a single Fourier mode with wave vector of magnitude $k=2\pi/\lambda$, where λ is the wavelength of the spatial perturbation. A formal generator for the dynamics of the response function is defined and expanded for long wavelengths to second order in k. If the coefficients in this expansion have a finite limit for long times, they define the corresponding transport matrix for the phenomenological equations. Accordingly, the parameters (cooling rate, pressure, and seven transport coefficients) are given definitions in terms of correlation functions for the reference HCS in this limit. There are two technical problems with this formal procedure. First, the "long-time" limit is complicated by the cooling dynamics of the reference HCS. This problem is solved here by treating the cooling as an independent degree of freedom, so that the limit is taken with respect to only the residual time dependence. The second problem is the possibility for many different formally equivalent forms for the correlation functions, resulting from different choices for the initial perturbation at the microscopic level. In general, the perturbations excite all dynamical responses in addition to hydrodynamics, and the limit of small k does not cleanly separate the two in all cases. Resulting expressions for macroscopic parameters generally include explicit complex transients that ultimately vanish in the long-time limit, but otherwise complicate their practical utility. This problem is eliminated for normal fluids by considering a special, physically relevant, initial perturbation in terms of the local conserved densities associated with the invariants of the dynamics. In this way, only the hydrodynamic excitations occur as $k\to 0$ and the usual "simple" Green-Kubo forms for transport coefficients are obtained. A similar choice is identified here for the same reasons, although the generators for the dynamics, the invariants, and local densities are all different for granular fluids.

The primary results reported in this paper can be summarized as follows. The pressure and the cooling rate that occur in the linearized hydrodynamic equations are identified as specific averages over the HCS. The transport coefficients associated with the heat and momentum fluxes and the cooling rate are identified as specific time correlation functions in the HCS. They are given in three equivalent representations, each of which is the tractable one for different analytical and numerical techniques that may be used for studying them further. They correspond to the three familiar representations for the diffusion coefficient of an impurity in a normal fluid,

$$D=-\lim\frac{1}{d}\frac{\partial}{\partial t}\langle \boldsymbol{q}_0\cdot\boldsymbol{q}_0(t)\rangle_e=-\lim\frac{1}{d}\langle \boldsymbol{q}_0\cdot\boldsymbol{v}_0(t)\rangle_e$$

$$=\lim\frac{1}{d}\int_0^t dt'\langle \boldsymbol{v}_0\cdot\boldsymbol{v}_0(t')\rangle_e, \quad (2)$$

where $\boldsymbol{q}_0$ and $\boldsymbol{v}_0$ are the position and velocity of the impurity, d is the dimension of the system, the angular brackets denote equilibrium averages, and lim means the usual thermodynamic limit of large volume V and number of particles N, followed by large t. The first expression is the Einstein relationship, indicating that the mean square displacement grows linearly with time for long times. It was subsequently generalized to other transport coefficients by Helfand [14], and it is usually referred to as the Einstein-Helfand representation. The other two representations are the intermediate Helfand and Green-Kubo forms, respectively and this is the nomenclature used for all the transport coefficients in the rest of this paper.

The derivation of these formal results is accomplished with few restrictions on the dynamics in phase space: deterministic, Markovian, and invertible. This allows a wide range of inelastic collision rules currently used for granular fluids, from inelastic hard spheres to soft viscoelastic potentials. However, to expose further details of the expressions for the transport coefficients obtained here, and to provide insight into the content of these formal expressions in a more tractable setting, a companion paper following this presentation is specialized to the simplest case of smooth inelastic hard spheres with constant coefficient of restitution [18].

The structure of the paper is as follows. In Sec. II, the phenomenological hydrodynamic equations associated with a granular fluid are described and its homogeneous solution characterized. These equations are linearized about the homogeneous state and the first of two representations of the hydrodynamic response matrix identified. Also, the technical complications associated with the fact that the homogeneous reference state is time dependent are circumvented by treating the HCS temperature as an independent variable, separating the reference cooling dynamics from the residual time dependence. Then, in Sec. III, the tools of statistical mechanics (e.g., the generators of dynamics of phase functions and distribution functions, the Liouville equation, and time correlation functions) are described. The HCS phase space distribution is characterized as a special solution to the Liouville equation. Next, for more general states, the HCS cooling and residual time dependence are again distinguished using the HCS temperature as an independent variable. The resulting Liouville equation in this representation has the HCS as a stationary solution. The invariants of the generator for the Liouville dynamics are obtained from this stationary condition. A class of homogeneous solutions to the Liouville equation is constructed from these invariants and shown to represent excitations that agree with those from hydrodynamics in the long-wavelength limit. This provides the necessary identification of special initial perturbations for the linear response analysis that excite only hydrodynamics in the long-wavelength limit and avoid the complex initial transients mentioned above.

In Sec. IV, the linear response problem is formulated in the context of statistical mechanics and the second representation for the hydrodynamic response matrix in terms of time correlation functions over the HCS is obtained. In Sec. V, the long-wavelength limit of the general hydrodynamic response function is described in order to obtain Navier-Stokes order parameters. The primary results of the paper are presented and discussed in Secs. VI and VII. In Sec. VIII, a dimensionless representation of the dynamics of a granular fluid is developed with explicit identification of the energy scales in the problem to expose the simplifying features of hard spheres or disks, reported in the following companion paper. Finally, Sec. IX is used to discuss the context and scope of this work. Due to the involved nature of the analysis done, many details are deferred to one of six Appendixes included in this presentation.

II. PHENOMENOLOGICAL HYDRODYNAMICS

The purpose of this section is twofold. First, the phenomenological hydrodynamic equations are recalled and the unknown parameters (pressure, cooling rate, transport coefficients) are indicated. A special solution for spatially homogeneous states is obtained, and the hydrodynamic equations are linearized about that state for small spatial perturbations to describe linear response. The second purpose is to characterize the dynamics to be expected from solutions to these equations. As it will be shown below, two features new to granular fluids are an inherent time dependence of the coefficients due to the cooling of the reference state, and a nontrivial dynamics associated with homogeneous perturbations of the homogeneous state. In the subsequent sections, it will be shown that identifying the component of linear response associated with only spatial perturbations requires taking explicit account of these two types of homogeneous dynamics.

Hydrodynamics for a one-component fluid is a closed description for the dynamics in terms of the number density $n(\boldsymbol{r},t)$, the energy density $e(\boldsymbol{r},t)$, and the momentum density $\boldsymbol{g}(\boldsymbol{r},t)$. The starting point for identifying such a description for a granular fluid is the formally exact macroscopic balance equations for these densities. However, as for normal fluids, it is usual to replace the energy density and momentum density by the temperature $T(\boldsymbol{r},t)$ and flow velocity $\boldsymbol{U}(\boldsymbol{r},t)$ as the hydrodynamic variables, together with the number density. This is accomplished through the definitions

$$e(\boldsymbol{r},t) \equiv \frac{1}{2}mn(\boldsymbol{r},t)U^2(\boldsymbol{r},t) + e_0[n(\boldsymbol{r},t),T(\boldsymbol{r},t)], \tag{3}$$

$$\boldsymbol{g}(\boldsymbol{r},t) \equiv mn(\boldsymbol{r},t)\boldsymbol{U}(\boldsymbol{r},t). \tag{4}$$

Here m is the mass of a particle and $e_0(n,T)$ is some specified function of n and T. The two most common choices are $e_0(n,T)=dnT/2$, where d is the dimension of the system, or $e_0(n,T)=e_e(n,T)$, the thermodynamic function for the corresponding equilibrium fluid. The former is common in applications of computer simulations (note that the Boltzmann constant has been set equal to unity), while the latter is the historical choice in most formulations of hydrodynamics. Both definitions coincide for the special case of hard spheres. For normal and also for granular fluids, the choice made constitutes a *definition* of temperature for nonequilibrium states and has no *a priori* thermodynamic implications. The macroscopic balance equations in terms of n, T, and $\boldsymbol{U}$ are

$$D_t n + n \boldsymbol{\nabla} \cdot \boldsymbol{U} = 0, \tag{5}$$

$$D_t U_i + (mn)^{-1}\frac{\partial}{\partial r_j}P_{ij} = 0, \tag{6}$$

$$\left(\frac{\partial e_0}{\partial T}\right)_n (D_t + \zeta)T + \left[e_0 - n\left(\frac{\partial e_0}{\partial n}\right)_T\right] \boldsymbol{\nabla} \cdot \boldsymbol{U} + P_{ij}\frac{\partial U_i}{\partial r_j} + \boldsymbol{\nabla} \cdot q = 0, \tag{7}$$

where $D_t \equiv \partial_t + U \cdot \boldsymbol{\nabla}$ is the material derivative, ζ is the cooling rate due to the energy loss from the interaction between granular particles, q is the heat flux, and P_{ij} is the pressure tensor. These equations have the same form as those for a normal fluid, except for the presence of the term involving the cooling rate ζ in the equation for the temperature.

The above balance equations are not a closed set of equations until $\boldsymbol{q}$, P_{ij}, and ζ are specified as functionals of the hydrodynamic fields, i.e., their space and time dependence occurs entirely through these fields. This happens for normal fluids on length and time scales long compared to the mean free path and mean free time, respectively, and similar conditions may be assumed for granular fluids as well. If, furthermore, the state of the system is such that the spatial variation of the fields is smooth, then an expansion of these functionals in gradients of the fields can be performed. The coefficients in this expansion further simplify from fluid isotropy and the results to first order in the gradients must have the form

$$P_{ij} = p(n,T)\delta_{ij} - \eta(n,T)\left(\frac{\partial U_i}{\partial r_j} + \frac{\partial U_j}{\partial r_i} - \frac{2}{d}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{U}\right) - \kappa(n,T)\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{U} + \ldots; , \tag{8}$$

$$\boldsymbol{q} = -\lambda(n,T)\boldsymbol{\nabla}T - \mu(n,T)\boldsymbol{\nabla}n + \ldots; , \tag{9}$$

$$\zeta = \zeta_0(n,T) + \zeta^U(n,T)\boldsymbol{\nabla}\cdot\boldsymbol{U} + \zeta^T(n,T)\nabla^2 T + \zeta^n(n,T)\nabla^2 n + \ldots; , \tag{10}$$

where δ_{ij} is the Kronecker delta symbol and the ellipses denote terms proportional to higher powers or higher-degree spatial derivatives of the hydrodynamic fields than those written explicitly. These expressions represent the "constitutive equations" which, together with the macroscopic balance equations, give Navier-Stokes order hydrodynamics for a granular fluid. Note that the cooling rate is required to second order in the gradients, while the pressure tensor and heat flux are required only to first order (since only their gradients occur in the balance equations). The pressure tensor has the same form as Newton's viscosity law for a normal fluid, while the expression for the heat flux is a generalization of Fourier's law [19,20].

The expressions (8)–(10) include unspecified functions: the pressure $p(n,T)$, the zeroth order in the gradients cooling rate $\zeta_0(n,T)$, the transport coefficients from the cooling rate $\zeta^U(n,T)$, $\zeta^T(n,T)$, $\zeta^n(n,T)$, the shear viscosity $\eta(n,T)$, the bulk viscosity $\kappa(n,T)$, the thermal conductivity $\lambda(n,T)$, and a new heat flux coefficient $\mu(n,T)$. All of these must be provided by experiment or by the theoretical justification of this phenomenology.

Although the Navier-Stokes equations are based on the small-gradient forms for the constitutive equations, it does not mean that they are limited to systems close to a global homogeneous state. Thus, they can be applicable locally over domains larger than the mean free path, but the hydrodynamic fields may still vary significantly throughout the system. Consequently, a wide range of experimental and simulation conditions for granular fluids have been well-described by the Navier-Stokes equations [21–24].

The spatially homogeneous solution to Eqs. (5)–(7) for an isolated system (e.g., periodic boundary conditions) is

$$n(\boldsymbol{r},t) = n_h, \quad \boldsymbol{U}(\boldsymbol{r},t) = \boldsymbol{U}_h, \quad T(\boldsymbol{r},t) = T_h(t), \tag{11}$$

where $T_h(t)$ is the solution to

$$\{\partial_t + \zeta_0[n_h,T_h(t)]\}T_h(t) = 0. \tag{12}$$

All time dependence of this state occurs through the homogeneous temperature $T_h(t)$, so this is referred to as the homogeneous cooling state [25]. Once the functional form for $\zeta_0[n_h,T_h(t)]$ has been established, the first-order nonlinear equation (12) can be solved by direct integration for a given initial condition.

Now consider small spatial perturbations of the HCS, assuming that $T_h(t)$ is known,

$$y_\beta(\boldsymbol{r},t) = y_{\beta,h} + \delta y_\beta(\boldsymbol{r},t), \quad \{y_\beta\} \equiv \{n,T,\boldsymbol{U}\}, \tag{13}$$

with $\delta y_\beta(\mathbf{r},t)$ sufficiently small that nonlinear terms in the hydrodynamic equations can be neglected. The resulting linear equations have coefficients independent of $\boldsymbol{r}$ so the differential equations can be given an algebraic representation by means of the Fourier transformation,

$$\delta\tilde{y}_\beta(\boldsymbol{k},t) = \int d\boldsymbol{r}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta y_\beta(\boldsymbol{r},t). \tag{14}$$

Moreover, the components of the flow velocity are separated into a longitudinal component relative to $\boldsymbol{k}$, $\delta\tilde{U}_\parallel = \hat{\boldsymbol{k}}\cdot\delta\tilde{\boldsymbol{U}}$, and $d-1$ transverse components $\delta\tilde{U}_{\perp,i} = \hat{\boldsymbol{e}}_i\cdot\delta\tilde{\boldsymbol{U}}$, where $\hat{\boldsymbol{k}} \equiv \boldsymbol{k}/k$ and $\{\hat{\boldsymbol{e}}_i;\ i=1,\ldots,d-1\}$ are a set of d pairwise orthogonal unit vectors. Therefore, in the following

$$\{\tilde{y}_\beta\} \equiv \{\tilde{n},\tilde{T},\tilde{U}_\parallel,\tilde{\boldsymbol{U}}_\perp\}, \tag{15}$$

with $\tilde{\boldsymbol{U}}_\perp$ denoting the set of the $d-1$ components $\tilde{U}_{\perp,i}$. The linearized hydrodynamic equations have the form

$$\{\partial_t + \mathcal{K}^{\text{hyd}}[n_h,T_h(t);\boldsymbol{k}]\}\delta\tilde{y}(\boldsymbol{k},t) = 0, \tag{16}$$

where a $(d+2)$-dimensional matrix notation has been introduced for simplicity. The transport matrix $\mathcal{K}^{\text{hyd}}$ is block diagonal, with decoupled longitudinal and transverse submatrices,

$$\mathcal{K}^{\text{hyd}} = \begin{pmatrix} \mathcal{K}_1^{\text{hyd}} & 0 \\ 0 & \mathcal{K}_2^{\text{hyd}} \end{pmatrix}, \tag{17}$$

$$\mathcal{K}_1^{\text{hyd}} = \begin{pmatrix} 0 & 0 & -in_h k \\ \dfrac{\partial(\zeta_0 T_h)}{\partial n_h} + \left(\dfrac{\mu}{e_{0,T}} - \zeta^n T_h\right)k^2 & \dfrac{\partial(\zeta_0 T_h)}{\partial T_h} + \left(\dfrac{\lambda}{e_{0,T}} - \zeta^T T_h\right)k^2 & -i\left(\zeta^U T_h + \dfrac{h - e_{0,n}n_h}{e_{0,T}}\right)k \\ -\dfrac{ip_n k}{n_h m} & -\dfrac{ip_T k}{n_h m} & \dfrac{1}{n_h m}\left(\dfrac{2(d-1)}{d}\eta + \kappa\right)k^2 \end{pmatrix}, \tag{18}$$

$$\mathcal{K}_2^{\mathrm{hyd}} = \frac{\eta}{n_h m} k^2 I. \qquad (19)$$

In the above expressions, I is the unit matrix of dimension $d-1$, $h \equiv e_0 + p$, and

$$e_{0,n} \equiv \left(\frac{\partial e_0}{\partial n}\right)_T, \quad e_{0,T} \equiv \left(\frac{\partial e_0}{\partial T}\right)_n, \quad p_n \equiv \left(\frac{\partial p}{\partial n}\right)_T,$$

$$p_T \equiv \left(\frac{\partial p}{\partial T}\right)_n. \qquad (20)$$

It is understood in Eqs. (18) and (19) that all the quantities are to be evaluated at $n = n_h$ and $T = T_h(t)$.

There are $d+2$ independent fundamental solutions for Eqs. (16) in terms of which the general solution to the initial value problem can be expressed. These are the hydrodynamic modes. For normal fluids, they are simply the eigenfunctions of the matrix $\mathcal{K}^{\mathrm{hyd}}$ and the hydrodynamic excitations are exponential in time with the corresponding eigenvalues. Here, the identification is somewhat less direct due to the dependence of $\mathcal{K}^{\mathrm{hyd}}$ on $T_h(t)$. For example, the shear modes are proportional to

$$\exp\left(-\int^t dt' \frac{\eta[n_h, T_h(t')]}{n_h m} k^2\right), \qquad (21)$$

and their time dependence is no longer exponential in time.

The general solution to the initial value problem can be written in terms of a matrix of response functions (Green's function matrix) corresponding to the representation in Eq. (1),

$$\delta \tilde{y}(\boldsymbol{k}, t) = \tilde{C}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}, t] \delta \tilde{y}(\boldsymbol{k}, 0), \qquad (22)$$

which is determined from the linearized hydrodynamic equations,

$$\{\partial_t + \mathcal{K}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}]\} \tilde{C}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}, t] = 0, \qquad (23)$$

$$\tilde{C}_{\alpha\beta}^{\mathrm{hyd}}[n_h, T_h(0); \boldsymbol{k}, 0] = \delta_{\alpha\beta}. \qquad (24)$$

Conversely, if $\tilde{C}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}, t]$ were given, the parameters of the transport matrix $\mathcal{K}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}]$ could be determined from a rearrangement of (23),

$$\mathcal{K}^{hyd}[n_h, T_h(t); \boldsymbol{k}] = -\{\partial_t \tilde{C}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}, t]\} \times \tilde{C}^{\text{hyd-1}}[n_h, T_h(t); \boldsymbol{k}, t]. \qquad (25)$$

Equations (22) and (25) suggest the formally exact approach to be developed in the next sections to identify the parameters of $\mathcal{K}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}]$. First, a general response function $\tilde{C}$ is defined in place of $\tilde{C}^{\mathrm{hyd}}$ for $\delta \tilde{y}(\boldsymbol{k}, t)$. Next, a transport matrix is defined as in Eq. (25) with $\tilde{C}^{\mathrm{hyd}}$ replaced by $\tilde{C}$. If the phenomenological description exists on long time and small $\boldsymbol{k}$ scales, then this expression becomes $\mathcal{K}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}]$, providing a formal definition of the hydrodynamic parameters. For normal fluids, this procedure gives the familiar Green-Kubo expressions for the transport coefficients in terms of time correlation functions in the equilibrium reference state [15].

Implementation of this approach for granular fluids requires addressing two new features not present for normal fluids. The first is the time dependence of the reference state and the consequent parametrization of $\mathcal{K}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}]$ by the time-dependent temperature $T_h(t)$ [e.g., the transport coefficients are functions of $T_h(t)$]. It is useful to suppress this dynamics by the definitions

$$\mathcal{K}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}] = [\mathcal{K}^{\mathrm{hyd}}(n, T; \boldsymbol{k})]_{n=n_h, T=T_h(t)},$$

$$\tilde{C}^{\mathrm{hyd}}[n_h, T_h(t); \boldsymbol{k}, t] = [\tilde{C}^{\mathrm{hyd}}(n, T; \boldsymbol{k}, t)]_{n=n_h, T=T_h(t)}. \qquad (26)$$

Thus $\mathcal{K}^{\mathrm{hyd}}(n, T; \boldsymbol{k})$ and $\tilde{C}^{\mathrm{hyd}}(n, T; \boldsymbol{k}, t)$ are the universal functions associated with a general class of reference states, and they reduce to the hydrodynamic response functions characterizing the perturbation about the HCS only when T is related to t through the particular solution for $T_h(t)$ from (12). In this way, t and T are treated as independent variables so, for example, the equation for $\tilde{C}^{\mathrm{hyd}}(n, T; \boldsymbol{k}, t)$ is not Eq. (23) but

$$[\partial_t - \zeta_0(n, T) T \partial_T + \mathcal{K}^{\mathrm{hyd}}(n, T; \boldsymbol{k})] \tilde{C}^{\mathrm{hyd}}(n, T; \boldsymbol{k}, t) = 0, \qquad (27)$$

with the initial condition $\tilde{C}_{\alpha\beta}^{\mathrm{hyd}}(n, T; \boldsymbol{k}, 0) = \delta_{\alpha\beta}$. The time derivative ∂_t is now at constant T. The new term with the temperature derivative in Eq. (27) is the generator for the dynamics of $T_h(t)$ in the sense that, for any arbitrary function $X(T)$,

$$X[T_h(t; T)] = \exp[-t \zeta_0(n_h, T) T \partial_T] X(T), \qquad (28)$$

where $T_h(t; T)$ is the solution to Eq. (12) with $T_h(0) = T$ (the proof is given in Appendix A). Equation (27) now has time-independent coefficients, but at the price of introducing a new independent variable T and a new generator for the dynamics. Still, it is most convenient to work with the generic forms $\mathcal{K}^{\mathrm{hyd}}(n, T; \boldsymbol{k})$ and $\tilde{C}^{\mathrm{hyd}}(n, T; \boldsymbol{k}, t)$ and the counterpart of Eq. (25),

$$\mathcal{K}^{\mathrm{hyd}}(n, T; \boldsymbol{k}) = -\{[\partial_t - \zeta_0(n, T) T \partial_T] \tilde{C}^{\mathrm{hyd}}(n, T; \boldsymbol{k}, t)\} \tilde{C}^{\mathrm{hyd}-1} \times (n, T; \boldsymbol{k}, t). \qquad (29)$$

The above procedure is adopted at the level of statistical mechanics in the next section as well, to distinguish the time dependence of the reference state from all residual time dependence.

The second new feature is the existence of a nonzero generator for homogeneous (zero-wave-vector) dynamics, i.e., $\mathcal{K}^{\mathrm{hyd}}(n, T; \boldsymbol{0}) \neq 0$,

$$\mathcal{K}_1^{\mathrm{hyd}}(n, T; \boldsymbol{0}) = \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{\partial(\zeta_0 T)}{\partial n} & \dfrac{\partial(\zeta_0 T)}{\partial T} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{K}_2^{\mathrm{hyd}} = 0, \qquad (30)$$

and consequently $\tilde{C}_{\alpha\beta}^{\mathrm{hyd}}(n, T; \boldsymbol{0}, t) \neq \delta_{\alpha\beta}$ (see Appendix A):

$$\widetilde{C}^{\text{hyd}}(n,T;\mathbf{0},t)=\begin{pmatrix}\widetilde{C}_1^{\text{hyd}} & 0\\ 0 & I\end{pmatrix}, \tag{31}$$

where

$$\widetilde{C}_1^{\text{hyd}}(n,T;\mathbf{0},t)=\begin{pmatrix}1 & 0 & 0\\ \left(\frac{\partial T}{\partial n}\right)_{T(-t;T)} & \left(\frac{\partial T}{\partial T(-t;T)}\right)_n & 0\\ 0 & 0 & 1\end{pmatrix} \tag{32}$$

and I is again the unit matrix of dimension $d-1$. The interpretation of the above result is clear when evaluated at $n=n_h$, $T=T_h(t)$, where the nontrivial matrix elements become

$$\widetilde{C}_{21}^{\text{hyd}}[n_h,T_h(t);\mathbf{0},t]=\left(\frac{\partial T_h(t)}{\partial n_h}\right)_{T_h(0)},$$

$$\widetilde{C}_{22}^{\text{hyd}}[n_h,T_h(t);\mathbf{0},t]=\left(\frac{\partial T_h(t)}{\partial T_h(0)}\right)_{n_h}. \tag{33}$$

This is just the linear response of the solution to Eq. (12) due to changes in the initial conditions. It is the additional dynamics of the temperature beyond that of $T_h(t)$, due to initial homogeneous density and temperature perturbations, as the system attempts to approach a new HCS. It is a hydrodynamic excitation, and it is expected to be distinct from the rapid homogeneous relaxation of other microscopic modes on a much shorter time scale outside the scope of hydrodynamics. Hence for granular fluids the hydrodynamic modes cannot be identified simply as those that vanish for $k\to 0$, as for a normal fluid. Instead, they are identified as those modes whose homogeneous dynamics becomes that of Eq. (31) for $k\to 0$. Their identification at the level of statistical mechanics is a necessary first step before the linear response analysis can be carried out. This is done in the next section.

III. STATISTICAL MECHANICS

In this section, tools are developed to formulate the linear response of the fluid at the microscopic level. First, the dynamics of observables (phase functions) and states (phase space densities) is described. The equation for the states is the granular Liouville equation. A special solution to the Liouville equation corresponding to the macroscopic HCS of the previous section is characterized. The Liouville equation for general states is then rewritten to distinguish HCS cooling from all residual dynamics. In this representation, the HCS is a stationary state. The invariants of the associated generator for the Liouville equation are identified, and a special class of homogeneous solutions is constructed from them. Finally, it is shown that these solutions correspond to the hydrodynamic homogeneous response (31) and (32), i.e., the long-wavelength hydrodynamic excitations observed at the microscopic level of the Liouville equation. These provide the necessary ingredients to formulate the appropriate linear response problem and to identify the corresponding formally exact response functions $\widetilde{C}_{\alpha\beta}(n,T;\boldsymbol{k},t)$ in Sec. IV.

Consider a volume V enclosing N particles that interact and lose energy as a result of this interaction. Also suppose that the interactions are specified in such a way that if the positions and velocities of each of the particles are given at a time t_0, then there exists a well-defined trajectory for the evolution of the system for all times both earlier and later than t_0. The microscopic initial state of the system is entirely determined by the positions and velocities of all particles, denoted by a point in a $2dN$-dimensional phase space $\Gamma\equiv\{\boldsymbol{q}_r,\boldsymbol{v}_r;\ r=1,\ldots,N\}$. The state of the system at a later time t is completely characterized by the positions and velocities of all particles at that time $\Gamma_t\equiv\{\boldsymbol{q}_r(t),\boldsymbol{v}_r(t);\ r=1,\ldots,N\}$. In summary, the dynamics is Markovian and invertible.

The statistical mechanics for this system is comprised of the dynamics just described, a macrostate specified in terms of a probability density $\rho(\Gamma)$, and a set of observables (measurables). The expectation value for an observable A at time $t>0$, given a state $\rho(\Gamma)$ at $t=0$, is defined by

$$\langle A(t);0\rangle\equiv\int d\Gamma\ \rho(\Gamma)A(\Gamma_t). \tag{34}$$

The notation $A(\Gamma_t)$ indicates the function of a given phase point shifted forward in time along a trajectory. Equivalently, this can be considered as a function of the initial phase point and the time, $A(\Gamma,t)$. This second representation allows the introduction of a generator L for the time dependence defined by

$$\langle A(t);0\rangle=\int d\Gamma\ \rho(\Gamma)e^{tL}A(\Gamma), \tag{35}$$

where the explicit form of L is determined by the specific rule chosen for the interactions among the particles. This is left unspecified at the moment. Due to the assumption of invertibility made above, the dynamics can be transferred from the observable $A(\Gamma)$ to the state $\rho(\Gamma)$, by defining an adjoint generator $\bar{L}$,

$$\int d\Gamma\ \rho(\Gamma)e^{tL}A(\Gamma)\equiv\int d\Gamma[e^{-t\bar{L}}\rho(\Gamma)]A(\Gamma)$$
$$\equiv\int d\Gamma\ \rho(\Gamma,t)A(\Gamma). \tag{36}$$

The form generated by $\bar{L}$ is referred to as Liouville dynamics. This equivalence of observable and state dynamics is expressed in the above notation as

$$\langle A(t);0\rangle=\langle A;t\rangle. \tag{37}$$

In summary, the dynamics of phase functions is governed by an equation of the form

$$(\partial_t-L)A(\Gamma,t)=0, \tag{38}$$

and that of the probability distribution in phase space by a Liouville equation

$$(\partial_t+\bar{L})\rho(\Gamma,t)=0, \tag{39}$$

where L and $\bar{L}$ are time-independent operators.

Time correlation functions are defined in a similar way:

$$C_{AB}(t) \equiv \langle A(t)B(0);0\rangle \equiv \int d\Gamma\ \rho(\Gamma)A(\Gamma_t)B(\Gamma). \tag{40}$$

In terms of the generators introduced above, the correlation functions can be written

$$C_{AB}(t) = \int d\Gamma\ \rho(\Gamma)[e^{tL}A(\Gamma)]B(\Gamma), \tag{41}$$

or, equivalently,

$$C_{AB}(t) = \int d\Gamma\ A(\Gamma)e^{-t\bar{L}}[\rho(\Gamma)B(\Gamma)]. \tag{42}$$

Further comment on these generators and some examples are given in Appendix B.

A. Homogeneous reference state

In contrast to normal fluids, the average energy of the system decreases with time due to the nonconservative interactions among the particles,

$$\partial_t\langle E;t\rangle = \langle LE;t\rangle \leq 0, \tag{43}$$

where $E(\Gamma)$ is the phase function corresponding to the total energy and the second equality corresponds to the elastic limit. This loss of energy is a primary characteristic of granular fluids. It is convenient to introduce a granular temperature instead of the average energy in the same way as is done in (3) for the phenomenological equations and to characterize this energy loss as a "cooling" of the granular temperature. The temperature definition for a homogeneous state then is

$$e_0[n,T(t)] \equiv \frac{1}{V}\langle E;t\rangle - \frac{1}{2}mnU^2, \tag{44}$$

identified through the energy in the local rest frame, and Eq. (43) becomes

$$\partial_t T(t) = -\zeta(t)T(t), \tag{45}$$

where the "cooling rate" is identified as

$$\zeta(t) = -\frac{1}{VT}\left(\frac{\partial T}{\partial e_0}\right)_n \langle LE;t\rangle \geq 0. \tag{46}$$

The last inequality requires the physically relevant condition that $e_0(n,T)$ is an increasing function of T. This shows that there is no approach to equilibrium for a granular fluid, except in the elastic collision limit where $\zeta(t)=0$.

Consider first spatially homogeneous states. For normal fluids there is a rapid velocity relaxation, on the time scale of a few collisions per particle, after which the system is close to the Gibbs state. A similar behavior can be expected for a granular fluid, where the velocity relaxation leads to an analogous "universal" state which, however, is not stationary. Instead, universality suggests that its time dependence should not depend on the initial contitions but rather only through the global cooling temperature $T_h(t)$. Computer simulations support the existence of such a state [26], called the homogeneous cooling state in the following. It is the statistical ensemble underlying the macroscopic HCS of the previous section. Accordingly, the probability density for the HCS has the form [3,4]

$$\rho_h(\Gamma,t) = \rho_h[\Gamma;n_h,T_h(t),\boldsymbol{U}_h], \tag{47}$$

where n_h is the constant density, $T_h(t)$ the decreasing temperature, and $\boldsymbol{U}_h$ the uniform velocity flow. It follows that $\zeta_h(t)=\zeta_0[n_h,T_h(t)]$ is a function of time only through the temperature as well

$$\zeta_0(n,T) \equiv -\frac{1}{VT}\left(\frac{\partial T}{\partial e_0}\right)_n \int d\Gamma\ \rho_h(\Gamma;n,T)LE(\Gamma). \tag{48}$$

Substitution of this form into the Liouville equation gives the functional form of $\rho_h(\Gamma;n,T,\boldsymbol{U})$,

$$\bar{\mathcal{L}}_T\rho_h(\Gamma;n,T,\boldsymbol{U}) = 0, \tag{49}$$

with the definition

$$\bar{\mathcal{L}}_T \equiv -\zeta_0(n,T)T\partial_T + \bar{L}. \tag{50}$$

The time derivative has been evaluated using (45) for the HCS. Equations (48) and (49) constitute a pair of time-independent equations to determine both $\rho_h(\Gamma;n,T,U)$ and $\zeta_0(n,T)$. Once the latter is known, $T_h(t)$ is determined from the solution to Eq. (45),which becomes

$$\partial_t T_h(t) = -\zeta_0[n_h,T_h(t)]T_h(t). \tag{51}$$

Finally, $\rho_h(\Gamma;n,T,\boldsymbol{U})$ is also evaluated at $n=n_h$, $T=T_h(t)$, and $\boldsymbol{U}=\boldsymbol{U}_h$.

These results illustrate the leading nontrivial application of statistical mechanics to the derivation of a macroscopic dynamics: Eq. (51) is the long-wavelength hydrodynamics and its only parameter, $\zeta_0[n_h,T_h(t)]$, is now given a precise and formally exact definition, Eq. (48), from which to determine its density and temperature dependence.

It is easily verified that the equilibrium probability densities for normal fluids (e.g., Gibbs ensembles or maximum entropy ensembles for conserved densities) are not solutions to Eq. (49). Thus, the effects of inelastic collisions are twofold. First, they introduce an inherent time dependence due to energy loss, through $T_h(t)$, and second, they change the form of the probability density in a way that cannot be simply represented by the global invariants for a normal fluid. The HCS is defined by Eqs. (49) and (51) and no further consideration is given to its actual construction in the rest of this work. As noted above its existence is supported by molecular dynamics simulations that show a rapid approach to a state whose granular temperature obeys Eq. (51), with uniform density and flow velocity [26]. Mathematical proof of existence for solutions to the Liouville equation for any nonequilibrium state is rare. Here we assume that such a solution exists and use only those properties that are implied by Eq. (49). In the weak sense of observables calculated from this solution, generally much less detail is required (e.g., only the reduced distributions). This state is the reference state in terms of which the linear response properties are determined in the next sections.

B. Stationary representation and Invariants

Consider an arbitrary state $\rho(\Gamma,t)$, not necessarily homogeneous. There is a unique HCS state associated with it, determined by its initial total energy, particle number, and momentum, and consequently a corresponding function $T_h(t)$. From a purely formal point of view, the time dependence can be decomposed into a part represented by $T_h(t)$ and a residual time dependence

$$\rho(\Gamma;t)=\rho[\Gamma;n_h,T_h(t),\boldsymbol{U}_h;t]=[\rho(\Gamma;n,T,\boldsymbol{U};t)]_{n=n_h,T=T_h(t),\boldsymbol{U}=\boldsymbol{U}_h}. \tag{52}$$

The Liouville equation for $\rho(\Gamma;n,T,\boldsymbol{U};t)$ becomes

$$(\partial_t+\overline{\mathcal{L}}_T)\rho(\Gamma;n,T,\boldsymbol{U};t)=0. \tag{53}$$

It is again understood that t and T are now independent variables and the time derivative is taken at constant T. The new generator for the dynamics $\overline{\mathcal{L}}_T$ is given by Eq. (50), and includes the effect of the homogeneous cooling through the scaling operator $-\zeta_0(n,T)T\partial_T$. A primary motivation for this representation is the observation that the HCS $\rho_h(\Gamma;n,T,\boldsymbol{U})$ is a stationary solution to this Liouville equation, as shown by Eq. (49). Consequently, this will be referred to as the stationary representation. It is the most suitable one for studying excitations about the HCS and is the one used in the rest of this presentation.

For a normal fluid, the stationary solutions to the Liouville equation are functions of the global invariants. Conversely, these solutions can be viewed as generators for the invariants by suitable differentiation with respect to the ensemble parameters. The identification of the invariants for the granular fluid can be accomplished in the same way. To see this, differentiate the stationary condition (49) with respect to the hydrodynamic fields (11),

$$\frac{\partial}{\partial y_\alpha}\overline{\mathcal{L}}_T\rho_h(\Gamma;n,T,\boldsymbol{U})=0. \tag{54}$$

More explictly this is

$$\overline{\mathcal{L}}_T\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})=\Psi_\beta(\Gamma;n,T,\boldsymbol{U})\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0}), \tag{55}$$

where

$$\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})\equiv\left(\frac{\partial\rho_h(\Gamma;n,T,\boldsymbol{U})}{\partial y_\alpha}\right)_{\{y_\beta;\beta\neq\alpha\}}, \tag{56}$$

and $\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0})$ is the zero-wave-vector hydrodynamic transport matrix given by (30), now with the cooling rate ζ_0 given by (48). The functions Ψ_α are a set of $d+2$ phase space functions that define a subspace which is left invariant under the dynamics generated by $\overline{\mathcal{L}}_T$,

$$e^{-\overline{\mathcal{L}}_T t}\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})=\Psi_\beta(\Gamma;n,T,\boldsymbol{U})\widetilde{C}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0},t). \tag{57}$$

Here $\widetilde{C}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0},t)$ is the zero-wave-vector hydrodynamic response matrix of Eq. (31). This result follows directly from (55) and is proved in Appendix C. For this reason, the Ψ_α will be referred to as invariants.

It is significant that the Liouville dynamics on the left side of (57) becomes precisely the long-wavelength hydrodynamics in the coefficients of the right side. This can be interpreted further by considering an initial state for the Liouville equation of the form

$$\rho(\Gamma;0)=\rho_h(\Gamma;n,T,\boldsymbol{U})+\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})\delta y_\alpha(0). \tag{58}$$

This corresponds to a transformation of the HCS state by small homogeneous perturbations $\rho_h(\Gamma;n,T,\boldsymbol{U})\to\rho_h(\Gamma;n+\delta n,T+\delta T,\boldsymbol{U}+\delta\boldsymbol{U})$. The corresponding solution to the Liouville equation is

$$\begin{aligned}\rho(\Gamma,t)&=e^{-\overline{\mathcal{L}}_T t}[\rho_h(\Gamma;n,T,\boldsymbol{U})+\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})\delta y_\alpha(0)]\\&=\rho_h(\Gamma;n,T,\boldsymbol{U})+\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})\delta y_\alpha(t).\end{aligned} \tag{59}$$

where

$$\delta y_\alpha(t)=\widetilde{C}^{\text{hyd}}_{\alpha\beta}(n,T;\boldsymbol{0},t)\delta y_\beta(0). \tag{60}$$

The special choice of the Ψ_α for initial perturbations is seen to excite only hydrodynamic modes, and no other microscopic homogeneous excitations. In this sense, the $\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})$ can be considered the microscopic hydrodynamic modes in the long-wavelength limit. For a normal fluid, they become functions of the global invariants and therefore time independent, which are indeed the hydrodynamic excitations at $k=0$ in that case. For a granular fluid, as discussed in the previous section, there is a nontrivial dynamics even at $k=0$. The analogy can be made more direct by rewriting Eq. (57) in the form

$$\mathcal{U}_{\alpha\beta}(t,T)\Psi_\beta(\Gamma;n,T,\boldsymbol{U})=\Psi_\alpha(\Gamma;n,T,\boldsymbol{U}), \tag{61}$$

where the new matrix evolution operator $\mathcal{U}(t,T)$ is defined as

$$\mathcal{U}_{\alpha\beta}(t,T)\equiv\widetilde{C}^{\text{hyd}-1}_{\beta\alpha}(n,T;\boldsymbol{0},t)e^{-\overline{\mathcal{L}}_T t}. \tag{62}$$

The dynamics described by $\mathcal{U}(t,T)$ is the Liouville dynamics, compensated for both effects of cooling, through the generator in $\overline{\mathcal{L}}_T$, and the dynamics of homogeneous perturbations, through the response function $\widetilde{C}^{\text{hyd}}(n,T;\boldsymbol{0},t)$. Consequently, $\mathcal{U}(t,T)$ provides the dynamics associated with spatial perturbations. It will be seen below that this operator defines the time dependence of the correlation functions representing all transport coefficients. Equation (61) shows that the functions $\{\Psi_\alpha\}$ are global invariants for the generator of dynamics associated with spatial perturbations of this system and hence give a natural reference state about which to study the relaxation of spatial perturbations in this system, which in turn is the relevant object of study for the identification of hydrodynamic transport coefficients. Note that although it is convenient to keep $\boldsymbol{U}\neq\boldsymbol{0}$ at a formal level for the discussion here, there is no problem in taking the limit $\boldsymbol{U}\to\boldsymbol{0}$ in all the results obtained.

IV. INITIAL PREPARATION FOR GENERAL LINEAR RESPONSE

In this section, the response of the system to an initial spatial perturbation in the hydrodynamic fields relative to the HCS is studied, in order to extract the analog of the hydrodynamic transport matrix described on phenomenological grounds in Eqs. (17)–(19) above. The hydrodynamic deviations $\{\delta y_\alpha(\boldsymbol{r},t)\}=\{\delta n(\boldsymbol{r},t),\delta T(\boldsymbol{r},t),\delta \boldsymbol{U}(\boldsymbol{r},t)\}$ are obtained as phase space averages

$$\delta y_\alpha(\boldsymbol{r},t) = \int d\Gamma[\rho(\Gamma;t) - \rho_h(\Gamma;n,T,\boldsymbol{U})]a_\alpha(\Gamma;n,T;\boldsymbol{r}), \tag{63}$$

where the phase functions $\{a_\alpha(\Gamma;n,T;\boldsymbol{r})\}$ are linear combinations of the local number, energy, and momentum densities,

$$\{a_\alpha(\Gamma;n,T;\boldsymbol{r})\} \equiv \left\{\mathcal{N}(\Gamma;\boldsymbol{r}), \frac{1}{e_{0,T}}[\mathcal{E}(\Gamma;\boldsymbol{r}) - e_{0,n}\mathcal{N}(\Gamma;\boldsymbol{r})], \frac{\boldsymbol{\mathcal{G}}(\Gamma;\boldsymbol{r})}{nm}\right\}. \tag{64}$$

The microscopic number density $\mathcal{N}(\boldsymbol{r})$, energy density $\mathcal{E}(\boldsymbol{r})$, and momentum density $\boldsymbol{\mathcal{G}}(\boldsymbol{r})$ are given by

$$\begin{pmatrix} \mathcal{N}(\Gamma;\boldsymbol{r}) \\ \mathcal{E}(\Gamma;\boldsymbol{r}) \\ \boldsymbol{\mathcal{G}}(\Gamma;\boldsymbol{r}) \end{pmatrix} \equiv \sum_{r=1}^{N} \delta(\boldsymbol{r}-\boldsymbol{q}_r) \begin{pmatrix} 1 \\ \dfrac{mv_r^2}{2} + \dfrac{1}{2}\sum_{s\neq r} V(\boldsymbol{q}_{rs}) \\ m\boldsymbol{v}_r \end{pmatrix}. \tag{65}$$

In this expression, $\boldsymbol{q}_{rs}=\boldsymbol{q}_r-\boldsymbol{q}_s$ and $V(\boldsymbol{q}_{rs})$ is the pair potential for the conservative part of the interaction among particles, as discussed in Appendix B.

Consider an initial inhomogeneous perturbation of the HCS generated by the hydrodynamic fields $y_\alpha(\boldsymbol{r},0)$ of the form

$$\rho(\Gamma;0) = \rho_h(\Gamma;n,T) + \int d\boldsymbol{r}\; \psi_\alpha(\Gamma;n,T,\boldsymbol{U};\boldsymbol{r})\,\delta y_\alpha(\boldsymbol{r},0). \tag{66}$$

The phase functions $\{\psi_\alpha\}$ characterizing this perturbation are unspecified at this point (beyond normalization). The formal solution to the Liouville equation for this initial condition is

$$\rho(\Gamma;t) = \rho_h(\Gamma;n,T) + \int d\boldsymbol{r}\; e^{-\mathcal{L}_T t}\psi_\alpha(\Gamma;n,T,\boldsymbol{U};\boldsymbol{r})\,\delta y_\alpha(\boldsymbol{r},0). \tag{67}$$

The response in the hydrodynamic fields is therefore, in matrix notation,

$$\delta y(\boldsymbol{r},t) = \int d\boldsymbol{r}'\, C(n,T;\boldsymbol{r}-\boldsymbol{r}',t)\,\delta y(\boldsymbol{r}',0), \tag{68}$$

with the matrix of response functions defined by

$$C_{\alpha\beta}(n,T;\boldsymbol{r}-\boldsymbol{r}',t) = \int d\Gamma\; a_\alpha(\Gamma;n,T;\boldsymbol{r})e^{-\mathcal{L}_T t}\psi_\beta(\Gamma;n,T,\boldsymbol{U};\boldsymbol{r}'). \tag{69}$$

This notation expresses the translational invariance of the reference HCS and the generator for the dynamics. This is the general response function, representing all the possible excitations of the many-body dynamics. The corresponding Fourier representation is

$$\widetilde{C}_{\alpha\beta}(n,T;\boldsymbol{k},t) = V^{-1}\int d\Gamma\; \widetilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k})e^{-\mathcal{L}_T t}\widetilde{\psi}_\beta(\Gamma;n,T,\boldsymbol{U};-\boldsymbol{k}), \tag{70}$$

where $\widetilde{a}_\alpha$ and $\widetilde{\psi}_\beta$ are the Fourier transforms of the phase functions a_α and ψ_β, respectively. The choice of ψ_β is still arbitrary at this point except for the conditions for normalization of both $\rho(\Gamma;0)$ and $\rho_h(\Gamma;n,T)$, and the representation (63) for $\delta y_\alpha(\boldsymbol{r},0)$,

$$\int d\Gamma\; \psi_\alpha(\Gamma;n,T;\boldsymbol{r}) = 0,$$

$$\int d\Gamma\; a_\alpha(\Gamma;n,T;\boldsymbol{r})\psi_\beta(\Gamma;n,T,\boldsymbol{U};\boldsymbol{r}') = \delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{71}$$

The second relation above shows that $\{a_\beta\}$ and $\{\psi_\beta\}$ comprise biorthogonal sets.

To identify the macroscopic hydrodynamic equations, an equation for the response function in a form similar to (27) is written,

$$[\partial_t - \zeta_0(n,T)T\partial_T + \mathcal{K}(n,T;\boldsymbol{k},t)]\widetilde{C}(n,T;\boldsymbol{k},t) = 0,$$

$$\widetilde{C}_{\alpha\beta}(n,T;\boldsymbol{k},0) = \delta_{\alpha\beta}, \tag{72}$$

which provides a definition for the generalized transport matrix,

$$\begin{aligned}\mathcal{K}(n,T;\boldsymbol{k},t) &= -\{[\partial_t - \zeta_0(n,T)T\partial_T]\widetilde{C}(n,T;\boldsymbol{k},t)\}\widetilde{C}^{-1}(n,T;\boldsymbol{k},t)\\ &= \mathcal{K}^{\text{hyd}}(n,T;\boldsymbol{0}) - \{[\partial_t - \zeta_0(n,T)T\partial_T\\ &\quad + \mathcal{K}^{\text{hyd}}(n,T;\boldsymbol{0})]\widetilde{C}(n,T;\boldsymbol{k},t)\}\widetilde{C}^{-1}(n,T;\boldsymbol{k},t).\end{aligned} \tag{73}$$

In the second equality, the contribution from $\boldsymbol{k}=\boldsymbol{0}$ has been extracted explicitly, anticipating the expansion in k needed to obtain Navier-Stokes order hydrodynamics. This form also provides a motivation for an optimal choice of the initial perturbation. Consider the $\boldsymbol{k}=\boldsymbol{0}$ limit. In general the second term on the right side of (73) does not vanish, but instead represents initial transients due to nonhydrodynamic excitations which ultimately must vanish at long times. Such transients occur at each order in the k expansion leading to formally correct, but awkward, forms for the hydrodynamic parameters. This complication happens for both normal and granular fluids, and can be minimized by choosing a special initial perturbation such that the second term of (73) is identically zero at $\boldsymbol{k}=\boldsymbol{0}$. The possibility to do so is provided by identification of the invariants Ψ_α of the last section, which are shown by Eq. (57) to excite only the $\boldsymbol{k}=\boldsymbol{0}$ hydrodynamics. Therefore, the initial perturbation considered here is such that

$$\Psi_\alpha(\Gamma;n,T,\boldsymbol{U}) = \int d\boldsymbol{r}\ \psi_\alpha(\Gamma;n,T,\boldsymbol{U};\boldsymbol{r}) = \tilde{\psi}_\alpha(\Gamma;n,T,\boldsymbol{U};\boldsymbol{0}). \tag{74}$$

With this choice, the second term of (73) vanishes at $\boldsymbol{k}=\boldsymbol{0}$ and $\mathcal{K}(n,T;\boldsymbol{0},t)=\mathcal{K}^{\mathrm{hyd}}(n,T;\boldsymbol{0})$ for all times; there are no initial long-wavelength transients.

Equation (74) shows that the $\psi_\alpha(\Gamma;\boldsymbol{r})$ needed for the optimal initial perturbation are the densities associated with the invariants. To provide a more physical interpretation, write these densities in terms of a functional $\rho_{lh}[\Gamma|\{y_\beta\}]$,

$$\psi_\alpha(\Gamma;n,T,\boldsymbol{U};\boldsymbol{r}) = \left[\frac{\delta\rho_{lh}[\Gamma|\{y_\beta\}]}{\delta y_\alpha(\boldsymbol{r})}\right]_{\{y_\beta\}=\{n,T,\boldsymbol{U}\}}. \tag{75}$$

Then (74) becomes

$$\frac{\partial\rho_h(\Gamma;\{y_\beta\})}{\partial y_\alpha(\boldsymbol{r})} = \int d\boldsymbol{r}\left[\frac{\delta\rho_{lh}[\Gamma|\{y_\beta\}]}{\delta y_\alpha(\boldsymbol{r})}\right]_{\{y_\beta\}=\{n,T,\boldsymbol{U}\}}. \tag{76}$$

Therefore, the densities are generated from a *local* HCS distribution, $\rho_{lh}[\Gamma|\{y_\beta\}]$, generalizing the HCS function to a functional. This local HCS distribution is the analog of the local equilibrium distribution for a normal fluid. Qualitatively, it corresponds to the condition that each local domain has a HCS distribution characterized by the parameters $\{y_\alpha(\boldsymbol{r},0)=y_{\alpha,h}+\delta y_\alpha(\boldsymbol{r},0)\}$. The characterization as a local form for the HCS is provided by the conditions

$$\delta y_\alpha(\boldsymbol{r},0) = \int d\Gamma\{\rho_{lh}[\Gamma|\{y_\beta(0)\}]-\rho_h(\Gamma,n,T)\}a_\alpha(\Gamma;\{y_\beta\};\boldsymbol{r}), \tag{77}$$

$$[\rho_{lh}(\Gamma|\{y_\alpha\})]_{\{y_\alpha\}=\{y_{\alpha,h}\}} = \rho_h(\Gamma;\{y_{\alpha,h}\}), \tag{78}$$

$$\int d\boldsymbol{r}_1\ldots\int d\boldsymbol{r}_n\left[\frac{\delta^n\rho_{lh}}{\delta y_\alpha(\boldsymbol{r}_1)\cdots\delta y_\beta(\boldsymbol{r}_n)}\right]_{\{y_\alpha\}=\{y_{\alpha,h}\}} = \frac{\partial^n\rho_h(\Gamma;\{y_{\alpha,h}\})}{\partial y_{\alpha,h}\ldots\partial y_{\beta,h}}. \tag{79}$$

The first equality states that the local state has the correct average values for the $\{a_\alpha\}$; the others refer to the uniform limit (similar to constructing an analytic function from all of its derivatives at a point). This is sufficient for the conditions (71) and (74) to be satisfied. The response function (69) with this choice of initial preparation will be the primary object of study in all of the following. It now has the important property

$$\tilde{C}_{\alpha\beta}(n,T;\boldsymbol{0},t) = \tilde{C}^{\mathrm{hyd}}_{\alpha\beta}(n,T;\boldsymbol{0},t), \tag{80}$$

with $\tilde{C}^{\mathrm{hyd}}_{\alpha\beta}(n,T;\boldsymbol{0},t)$ given by Eq. (31). By construction, all microscopic homogeneous transients have been eliminated.

The full hydrodynamic matrix, as given by Eqs. (17)–(19), follows from the formal result (73) for small k (long wavelengths) and long times,

$$\mathcal{K}^{\mathrm{hyd}}(n,T;\boldsymbol{k}) \equiv \lim_{t\gg t_0,k\ll k_0}\mathcal{K}(n,T;\boldsymbol{k},t). \tag{81}$$

The characteristic time t_0 and wavelength k_0^{-1} are expected to be the mean free time and mean free path, respectively. Comparison of this expression with the form (17) provides a "derivation" of the linear hydrodynamic equations, and also gives the coefficients of those equations in terms of the response functions. A detailed comparison to order k^2 is the objective of the next few sections.

The result (73) is the first of three formal representations of the transport matrix to be obtained here. Its expansion to order k^2 leads directly to the Einstein-Helfand representation of the transport coefficients as the long-time limit of time derivatives of correlation functions. This is analogous to the diffusion coefficient D represented in terms of the time derivative of the mean square displacement, i.e., the first representation in Eq. (2) [27]. However, due to the homogeneous state dynamics, the relevant time derivative is $[\partial_t-\zeta_0(n,T)T\partial/\partial T+\mathcal{K}^{\mathrm{hyd}}(n,T;\boldsymbol{0})]$, so this form may not be optimal in practice. There is an intermediate Helfand form which entails correlation functions with nonzero long-time limits determining the transport coefficients. Finally, the third equivalent representation is the Green-Kubo form in terms of time integrals of correlation functions. These second and third forms are given in the next section and utilized to implement the k expansion.

V. NAVIER-STOKES HYDRODYNAMICS

The linearized Navier-Stokes equations follow from an evaluation of the transport matrix of Eq. (73) to order k^2. This can be accomplished by a direct expansion of $\tilde{C}_{\alpha\beta}(n,T;\boldsymbol{k},t)$ in powers of k [27]. It is somewhat more instructive to proceed in a different manner, using the microscopic conservation laws to expose the dominant k dependence. This allows interpretation of the phase functions occurring in the correlation functions of the final expressions.

A. Consequences of conservation laws

For normal fluids, the variables $\tilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k})$ are the Fourier transforms of linear combinations of the local conserved densities, so their time derivatives are equal to $i\boldsymbol{k}\cdot\tilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\boldsymbol{k})$, where the $\tilde{\boldsymbol{f}}_\alpha$ are the associated microscopic fluxes. The proportionality to k of the time derivatives means that they vanish in the long-wavelength limit, as appropriate for a conserved density. This allows evaluation of the time derivative in Eq. (73) and shows that the transport matrix is of order k. Then, shifting the time dependence to the other density in the response function and using again the conservation law, the dependence through order k^2 is exposed in terms of correlation functions involving the fluxes [15]. It is somewhat more complicated for granular fluids, although the general idea is the same.

Consider first the time derivative occurring in Eq. (73). In the following, the dependence on $\boldsymbol{U}$ of ψ_α and Ψ_α will be omitted in the notation, since $\boldsymbol{U}_h$ will be taken to vanish from here on. Using the definition of the matrix of response functions, Eq. (70), it can be transformed into

$$\left(\partial_t - \zeta_0(n,T)T\frac{\partial}{\partial T} + \mathcal{K}^{\text{hyd}}(n,T;\mathbf{0})\right)\widetilde{C}(n,T;\boldsymbol{k},t) = V^{-1}\int d\Gamma\left[\left(L - \zeta_0(n,T)T\frac{\partial}{\partial T} + \mathcal{K}^{\text{hyd}}(n,T;\mathbf{0})\right)\times\widetilde{a}(\Gamma;n,T;\boldsymbol{k})\right]e^{-\mathcal{L}_T t}\widetilde{\psi}(\Gamma;n,T;-\boldsymbol{k}), \tag{82}$$

where the adjoint generator L of $\bar{L}$ has been introduced. As mentioned above, for normal fluids $L\widetilde{a}_\alpha = i\boldsymbol{k}\cdot\widetilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\boldsymbol{k})$. Here, the inelastic collisions give rise to an additional energy loss $\widetilde{w}(\Gamma;\boldsymbol{k})$, which is not proportional to k and therefore cannot be absorbed in the flux. The new relationships are (see Appendix D)

$$L\widetilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k}) = i\boldsymbol{k}\cdot\widetilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\boldsymbol{k}) - \delta_{\alpha 2}\frac{\widetilde{w}(\Gamma;\boldsymbol{k})}{e_{0,T}}. \tag{83}$$

The detailed expressions for $\widetilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\boldsymbol{k})$ and $\widetilde{w}(\Gamma;\boldsymbol{k})$ are given in Appendix D. The prefactor $1/e_{0,T}$ in the energy loss term appears because of the definition of $\widetilde{a}_2$. For $\alpha \neq 2$, the right side gives the usual fluxes for number and momentum density. Inclusion of the additional terms $-\zeta_0(n,T)T\partial/\partial T + \mathcal{K}^{\text{hyd}}(n,T;\mathbf{0})$ in Eq. (82) modifies this result to

$$\left(L - \zeta_0(n,T)T\frac{\partial}{\partial T}\right)\widetilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k}) + \sum_\beta \mathcal{K}^{\text{hyd}}_{\alpha\beta}(n,T;\mathbf{0})\widetilde{a}_\beta(\Gamma;n,T;\boldsymbol{k}) = i\boldsymbol{k}\cdot\widetilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\boldsymbol{k}) - \delta_{\alpha 2}\widetilde{\ell}(\Gamma;n,T;\boldsymbol{k}), \tag{84}$$

where $\widetilde{\ell}(\Gamma;n,T;\boldsymbol{k})$ is defined by

$$\widetilde{\ell}(\Gamma;n,T;\boldsymbol{k}) \equiv \frac{1}{e_{0,T}}\left(\widetilde{w}(\Gamma;\boldsymbol{k}) - V^{-1}\sum_\alpha \widetilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k}) \times\int d\Gamma\,\Psi_\alpha(\Gamma;n,T)\widetilde{w}(\Gamma;\mathbf{0})\right). \tag{85}$$

The first contribution in this expression is the phase function whose average in the HCS gives the cooling rate,

$$\frac{1}{e_{0,T}}\int d\Gamma\,\rho_h(\Gamma;n,T)\widetilde{w}(\Gamma;\mathbf{0}) = \zeta_0(n,T)T. \tag{86}$$

The remaining terms assure that $\widetilde{\ell}(\Gamma;n,T;\mathbf{0})$ is orthogonal to the invariants, namely, that

$$\widetilde{\ell}(\Gamma;n,T;\mathbf{0}) = (1-P^\dagger)\frac{\widetilde{w}(\Gamma;\mathbf{0})}{e_{0,T}} = -(1-P^\dagger)\frac{LE(\Gamma)}{e_{0,T}}. \tag{87}$$

Here, $P^\dagger$ is the projection operator onto the set $\{\widetilde{a}_\alpha(\Gamma;n,T,\mathbf{0})\}$,

$$P^\dagger X(\Gamma) = V^{-1}\widetilde{a}_\alpha(\Gamma;n,T;\mathbf{0})\int d\Gamma\,\Psi_\alpha(\Gamma;n,T)X(\Gamma). \tag{88}$$

To verify that $P^\dagger$ is really a projection operator, recall that, as a consequence of Eq. (71), $\{\widetilde{a}_\alpha(\Gamma;n,T,;\mathbf{0})\}$ and $\{\Psi_\alpha(\Gamma;n,T)\}$ form a biorthogonal set,

$$V^{-1}\int d\Gamma\,\widetilde{a}_\alpha(\Gamma;n,T;\mathbf{0})\Psi_\beta(\Gamma;n,T) = \delta_{\alpha\beta}. \tag{89}$$

Use of Eq. (84) in Eq. (82) shows that the correlation functions $\widetilde{C}_{\alpha\beta}(n,T;\boldsymbol{k},t)$ obey the equations

$$\left[\partial_t - \zeta_0 T\frac{\partial}{\partial T} + \mathcal{K}^{\text{hyd}}(n,T;\mathbf{0})\right]\widetilde{C}(n,T;\boldsymbol{k},t) = i\boldsymbol{k}\cdot\widetilde{\boldsymbol{D}}(n,T;\boldsymbol{k},t) - \widetilde{S}(n,T;\boldsymbol{k},t). \tag{90}$$

The new correlation functions on the right-hand side, $\widetilde{\boldsymbol{D}}_{\alpha\beta}(n,T;\boldsymbol{k},t)$ and $\widetilde{S}_{\alpha\beta}(n,T;\boldsymbol{k},t)$, are similar to $\widetilde{C}_{\alpha\beta}(n,T;\boldsymbol{k},t)$ but with $\widetilde{a}_\alpha$ replaced by $\widetilde{\boldsymbol{f}}_\alpha$ and $\delta_{\alpha 2}\widetilde{\ell}$, respectively,

$$\widetilde{\boldsymbol{D}}_{\alpha\beta}(n,T;\boldsymbol{k},t) = V^{-1}\int d\Gamma\,\widetilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\boldsymbol{k})e^{-\mathcal{L}_T t}\widetilde{\psi}_\beta(\Gamma;n,T;-\boldsymbol{k}), \tag{91}$$

$$\widetilde{S}_{\alpha\beta}(n,T;\boldsymbol{k},t) = \delta_{\alpha 2}V^{-1}\int d\Gamma\,\widetilde{\ell}(\Gamma;n,T;\boldsymbol{k})e^{-\mathcal{L}_T t}\widetilde{\psi}_\beta(\Gamma;n,T;-\boldsymbol{k}). \tag{92}$$

The utility of Eq. (90) is that its application in Eq. (73) leads to an expression in which the transport matrix is exposed through first order in k,

$$\mathcal{K}(n,T;\boldsymbol{k},t) = \mathcal{K}^{\text{hyd}}(n,T;\mathbf{0}) - [i\boldsymbol{k}\cdot\widetilde{\boldsymbol{D}}(n,T;\boldsymbol{k},t) - \widetilde{S}(n,T;\boldsymbol{k},t)]\widetilde{C}^{-1}(n,T;\boldsymbol{k},t). \tag{93}$$

It follows from Eqs. (57) and (87) that $\widetilde{S}(n,T;0,t)=0$, so the term between square brackets on the right-hand side of Eq. (93) is at least of order k. However, this representation is still not optimal since the right-hand side has the homogeneous dynamics of $\widetilde{C}(n,T;\mathbf{0},t)$ that should be canceled. This technical point is addressed by introducing $\widetilde{C}^{-1}(n,T;\mathbf{0},t)$ in the evolution operator for the correlation functions to transform it to $\mathcal{U}_{\alpha\beta}(t,T)$, introduced in Eq. (62), i.e., defining

$$\widetilde{\psi}_\alpha(\Gamma;n,T;\boldsymbol{k},t) \equiv \mathcal{U}_{\alpha\beta}(t,T)\widetilde{\psi}_\beta(\Gamma;n,T;\boldsymbol{k}). \tag{94}$$

There is no $\boldsymbol{k}=0$ dynamics for $\widetilde{\psi}_\alpha(\Gamma;n,T;\boldsymbol{k},t)$ since $\widetilde{\psi}_\alpha(\Gamma;n,T;\mathbf{0})$ is an invariant. The transport matrix (93) then becomes

$$\mathcal{K}(n,T;\boldsymbol{k},t) = \mathcal{K}^{\text{hyd}}(n,T;\mathbf{0}) - [i\boldsymbol{k}\cdot\overline{\boldsymbol{D}}(n,T;\boldsymbol{k},t) - \overline{S}(n,T;\boldsymbol{k},t)]\overline{C}^{-1}(n,T;\boldsymbol{k},t). \tag{95}$$

The correlation functions with the overbar are the same as those with the tilde, except that now they are defined with the dynamics of (94),

$$\bar{C}_{\alpha\beta}(n,T;\boldsymbol{k},t) = V^{-1}\int d\Gamma\, \tilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k})\tilde{\psi}_\beta(\Gamma;n,T;-\boldsymbol{k},t), \tag{96}$$

$$\bar{\boldsymbol{D}}_{\alpha\beta}(n,T;\boldsymbol{k},t) = V^{-1}\int d\Gamma\, \tilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\boldsymbol{k})\tilde{\psi}_\beta(\Gamma;n,T;-\boldsymbol{k},t), \tag{97}$$

$$\bar{S}_{\alpha\beta}(n,T;\boldsymbol{k},t) = \delta_{\alpha 2}V^{-1}\int d\Gamma\, \tilde{\ell}(\Gamma;n,T;\mathbf{k})\tilde{\psi}_\beta(\Gamma;n,T;-\boldsymbol{k},t). \tag{98}$$

Equation (95) gives the *intermediate Helfand* representation referred to at the end of the last section. It has the advantage of being expressed in terms of the appropriate dynamics of $\mathcal{U}_{\alpha\beta}(t,T)$, as well as avoiding the complex time derivative of Eq. (73).

The equivalent Green-Kubo form is obtained by representing the correlation functions in Eq. (95) as time integrals. This is accomplished by observing that there are "conjugate" conservation laws associated with $\tilde{\psi}_\alpha(\Gamma;n,T;\boldsymbol{k},t)$. Their existence follows from the fact that the ψ_α's are the densities associated with the invariants Ψ_α. The conjugate conservation laws are

$$\partial_t\tilde{\psi}_\alpha(\Gamma;n,T;\boldsymbol{k},t) - i\boldsymbol{k}\cdot\tilde{\gamma}_\alpha(\Gamma;n,T;\boldsymbol{k},t) = 0. \tag{99}$$

The new fluxes $\tilde{\gamma}_\alpha$ are identified in Appendix D as

$$\tilde{\gamma}_\alpha(\Gamma;n,T;\boldsymbol{k},t) = \mathcal{U}_{\alpha\beta}(t,T)\tilde{\gamma}_\beta(\Gamma;n,T;\boldsymbol{k}) \tag{100}$$

with

$$i\mathbf{k}\cdot\tilde{\gamma}_\alpha(\Gamma;n,T;\boldsymbol{k}) \equiv -\bar{\mathcal{L}}_T\tilde{\psi}_\alpha(\Gamma;n,T;\boldsymbol{k}) + \mathcal{K}^{\rm hyd}_{\beta\alpha}(n,T;\mathbf{0})\tilde{\psi}_\beta(\Gamma;n,T;\boldsymbol{k}). \tag{101}$$

These new conservation laws give directly

$$\partial_t\bar{C}_{\alpha\beta}(n,T;\boldsymbol{k},t) + i\boldsymbol{k}\cdot\bar{\boldsymbol{E}}_{\alpha\beta}(n,T;\boldsymbol{k},t) = 0, \tag{102}$$

$$\partial_t\bar{\boldsymbol{D}}_{\alpha\beta}(n,T;\boldsymbol{k},t) + i\boldsymbol{k}\cdot\bar{\mathsf{F}}_{\alpha\beta}(n,T;\boldsymbol{k},t) = 0, \tag{103}$$

$$\partial_t\bar{S}_{\alpha\beta}(n,T;\boldsymbol{k},t) + i\boldsymbol{k}\cdot\bar{\boldsymbol{N}}_{\alpha\beta}(n,T;\boldsymbol{k},t) = 0, \tag{104}$$

where

$$\bar{\boldsymbol{E}}_{\alpha\beta}(n,T;\boldsymbol{k},t) = V^{-1}\int d\Gamma\, \tilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k})\tilde{\gamma}_\beta(\Gamma;n,T;-\boldsymbol{k},t), \tag{105}$$

$$\bar{\mathsf{F}}_{\alpha\beta}(n,T;\boldsymbol{k},t) = V^{-1}\int d\Gamma\, \tilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\boldsymbol{k})\tilde{\gamma}_\beta(\Gamma;n,T;-\boldsymbol{k},t), \tag{106}$$

$$\bar{\boldsymbol{N}}_{\alpha\beta}(n,T;\boldsymbol{k},t) = \delta_{\alpha 2}V^{-1}\int d\Gamma\, \tilde{\ell}(\Gamma;n,T;\boldsymbol{k})\tilde{\gamma}_\beta(\Gamma;n,T;-\boldsymbol{k},t). \tag{107}$$

Note that $\bar{\mathsf{F}}_{\alpha\beta}$ is a second-rank tensor. Integrating Eqs. (102)–(104) allows $\bar{C}(n,T;\boldsymbol{k},t)$, $\bar{\boldsymbol{D}}(n,T;\boldsymbol{k},t)$, and $\bar{S}(n,T;\boldsymbol{k},t)$ to be eliminated from Eq. (95) in favor of $\bar{\boldsymbol{E}}$, $\bar{\mathsf{F}}$, and $\bar{\boldsymbol{N}}$, exposing a higher-order dependence on k,

$$\begin{aligned}\mathcal{K}(n,T;\boldsymbol{k},t) = \mathcal{K}^{\rm hyd}(n,T;\mathbf{0}) - \Bigg(& i\boldsymbol{k}\cdot\bar{\boldsymbol{D}}(n,T;\boldsymbol{k},0) - \bar{S}(n,T;\boldsymbol{k},0) \\ &+ i\boldsymbol{k}\int_0^t dt'\bar{\boldsymbol{N}}(n,T;\boldsymbol{k},t') + \boldsymbol{k}\boldsymbol{k}:\int_0^t dt'\bar{\mathsf{F}}(n,T;\boldsymbol{k},t')\Bigg) \\ &\times\left(I + i\boldsymbol{k}\cdot\int_0^t dt'\bar{C}^{-1}(n,T;\boldsymbol{k},t')\bar{\boldsymbol{E}}(n,T;\boldsymbol{k},t')\right. \\ &\left.\times\bar{C}^{-1}(n,T;\boldsymbol{k},t')\right).\end{aligned} \tag{108}$$

This is the *Green-Kubo* form for the transport matrix. An advantage of this form is a further exposure of the explicit dependence on k. In both Eqs. (95) and (108), the relevant correlation functions are seen to be those composed from the conserved densities $\{\tilde{a}_\alpha,\tilde{\psi}_\alpha\}$, the fluxes of the two kinds of conservation laws $\{\tilde{\boldsymbol{f}}_\alpha,\tilde{\gamma}_\alpha\}$, and the source term for inelastic collisions $\tilde{\ell}$. All of the time dependence is given by the evolution operator $\mathcal{U}_{\alpha\beta}(t,T)$ which is that for the N-particle motion in phase space, but compensated for all homogeneous dynamics.

B. Green-Kubo form to order k^2

Retaining only contributions to order k^2 in Eq. (108) gives

$$\begin{aligned}\mathcal{K}(n,T;\boldsymbol{k},t) = \mathcal{K}^{\rm hyd}(n,T;\mathbf{0}) - ik[\hat{\boldsymbol{k}}\cdot\bar{\boldsymbol{D}}(n,T;\mathbf{0},0) + \bar{Z}(n,T)] \\ + k^2[\Lambda(n,T) + \bar{Y}(n,T)].\end{aligned} \tag{109}$$

The terms first order in k on the right-hand side of this equation provide the parameters for Euler order hydrodynamics. At this order, the susceptibilities (pressure and pressure derivatives) are defined in terms of the time-independent correlation function $\bar{\boldsymbol{D}}_{\alpha\beta}(n,T;\mathbf{0},0)$, while the transport coefficient ζ^U is given by the Green-Kubo expression

$$\bar{Z}_{\alpha\beta}(n,T) = \delta_{\alpha 2}\delta_{\beta 3}T\zeta^U(n,T), \tag{110}$$

$$T\zeta^U(n,T) = -\hat{\boldsymbol{k}}\cdot\left(\bar{\boldsymbol{S}}^{(1)}_{23}(n,T;0) - \lim\int_0^t dt'\bar{\boldsymbol{N}}_{23}(n,T;\mathbf{0},t')\right). \tag{111}$$

The above identification has been made by comparison of the expression obtained here with the phenomenological transport matrix in Eqs. (17)–(19). Here and below, the notation for Taylor series expansion of any function $X(\boldsymbol{k})$ is

$$X(\boldsymbol{k}) = X(\mathbf{0}) + i\boldsymbol{k}\cdot\boldsymbol{X}^{(1)} - \boldsymbol{kk}:\mathsf{X}^{(2)} + \dots . \tag{112}$$

At order k^2, the Navier-Stokes transport coefficients in Eq. (109) are of two types. The first type are those obtained from $\Lambda(n,T)$, and they represent the dissipative contributions to the fluxes. They correspond to the shear and bulk viscosities, thermal conductivity, and μ coefficient in Eq. (9). In the Green-Kubo form they are determined by

$$\Lambda_{\alpha\beta}(n,T) = \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\overline{\mathsf{D}}^{(1)}_{\alpha\beta}(n,T;0) - \lim \int_0^t dt'\,\overline{\mathsf{G}}_{\alpha\beta}(n,T;t')\right), \tag{113}$$

with

$$\overline{\mathsf{G}}(n,T;t) = \overline{\mathsf{F}}(n,T;\mathbf{0},t) - \overline{\boldsymbol{D}}(n,T;\mathbf{0},0)\overline{\boldsymbol{E}}(n,T;\mathbf{0},t). \tag{114}$$

The transport coefficients of the second kind are those following from $\overline{Y}(n,T)$ in Eq. (109) and represent the second-order gradient contributions to the cooling rate, i.e., the coefficients ζ^T and ζ^n in Eq. (10),

$$\begin{aligned}\overline{Y}_{\alpha\beta}(n,T) = -\,\delta_{\alpha 2}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\Bigg(&\overline{\mathsf{S}}^{(2)}_{2\beta}(n,T;0)\\ &- \lim \int_0^t dt'\,\overline{\mathsf{H}}_{2\beta}(n,T;\mathbf{0},t')\Bigg),\end{aligned} \tag{115}$$

$$\overline{\mathsf{H}}_{2\beta}(n,T;\mathbf{0},t) = \overline{\mathsf{N}}^{(1)}_{2\beta}(n,T;t) + T\zeta^U(n,T)\hat{\boldsymbol{k}}\overline{\boldsymbol{E}}_{3\beta}(n,T;\mathbf{0},t). \tag{116}$$

The superscripts (1) and (2) denote coefficients in the expansion of correlation functions in powers of ik, as indicated in Eq. (112).

C. Intermediate Helfand form to order k^2

The intermediate Helfand form to order k^2 follows from direct expansion of Eq. (95). The structure is the same as in Eq. (119) as well as the contribution from $\overline{\boldsymbol{D}}_{\alpha\beta}(n,T;\mathbf{0},t) = \overline{\boldsymbol{D}}_{\alpha\beta}(n,T;\mathbf{0},0)$. On the other hand, the transport coefficients are now given by

$$T\zeta^U(n,T) = -\lim \hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{S}}^{(1)}_{23}(n,T;t), \tag{117}$$

$$\Lambda(n,T) = \lim \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:[\overline{\mathsf{D}}^{(1)}(n,T;t) - \overline{\boldsymbol{D}}(n,T;\mathbf{0},0)\overline{\boldsymbol{C}}^{(1)}(n,T;t)], \tag{118}$$

$$\begin{aligned}\overline{Y}_{\alpha\beta}(n,T) = -\,\delta_{\alpha 2}\lim[&\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\overline{\mathsf{S}}^{(2)}_{2\beta}(n,T;t) + T\zeta^U(n,T)\\ &\times\hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{C}}^{(1)}_{3\beta}(n,T;t)].\end{aligned} \tag{119}$$

The equivalence of these results with the Green-Kubo forms given in the previous subsection can be seen by noting that the conservation laws of Eqs. (102)–(104) to first order in k give

$$\overline{\boldsymbol{E}}(n,T;\mathbf{0},t) = -\,\partial_t\overline{\boldsymbol{C}}^{(1)}(n,T;t), \quad \overline{\mathsf{F}}(n,T;\mathbf{0},t) = -\,\partial_t\overline{\mathsf{D}}^{(1)}(n,T;t), \tag{120}$$

$$\overline{\boldsymbol{N}}(n,T;\mathbf{0},t) = -\,\partial_t\overline{\boldsymbol{S}}^{(1)}(n,T;t), \quad \overline{\mathsf{N}}^{(1)}(n,T;t) = -\,\partial_t\overline{\mathsf{S}}^{(2)}(n,T;t). \tag{121}$$

These allows the time integrals in the Green-Kubo expressions to be performed, giving directly Eqs. (117)–(119).

D. Einstein-Helfand form to order k^2

Finally, the Einstein-Helfand form to order k^2 follows from direct expansion of Eq. (73),

$$\mathcal{K}(n,T;\boldsymbol{k},t) = \mathcal{K}(n,T;\mathbf{0}) + i\boldsymbol{k}\cdot\boldsymbol{\mathcal{K}}^{(1)}(n,T;t) - \boldsymbol{kk}:\mathsf{K}^{(2)}(n,T;t), \tag{122}$$

with

$$\begin{aligned}\boldsymbol{\mathcal{K}}^{(1)}(n,T;t) = -\Bigg[&\left(\partial_t - \zeta_0(n,T)T\frac{\partial}{\partial T} + \mathcal{K}^{\text{hyd}}(n,T;\mathbf{0})\right)\\ &\times\widetilde{\boldsymbol{C}}^{(1)}(n,T;t)\Bigg]\widetilde{C}^{-1}(n,T;\mathbf{0};t)\end{aligned} \tag{123}$$

and

$$\begin{aligned}\mathsf{K}^{(2)}(n,T;t) = -\Bigg[&\left(\partial_t - \zeta_0(n,T)T\frac{\partial}{\partial T} + \mathcal{K}^{\text{hyd}}(n,T;\mathbf{0})\right)\\ &\times\widetilde{\mathsf{C}}^{(2)}(n,T;t) + \boldsymbol{\mathcal{K}}^{(1)}(n,T;t)\widetilde{\boldsymbol{C}}^{(1)}(n,T;t)\Bigg]\\ &\times\widetilde{C}^{-1}(n,T,\mathbf{0};t).\end{aligned} \tag{124}$$

This form does not separate explicitly the contributions leading to the transport coefficients at both Euler and Navier-Stokes orders for the cooling rate. In the previous two representations, this was possible because the microscopic phase function $\widetilde{\ell}(\Gamma;n,T;\boldsymbol{k})$, associated with collisional energy loss, appears explicitly.

E. Dynamics and projected fluxes

The Green-Kubo expressions involve the long-time limit of time integrals over correlation functions. This presumes that the correlation functions decay sufficiently fast for the integrals to exist. This decay time sets the time scale after which the hydrodynamic description can apply. If these integrals converge on that time scale then the Helfand formulas also reach their limiting plateau values on the same time scale.

To explore this time dependence further, consider the correlation function characterizing the transport coefficients $\Lambda_{\alpha\beta}(n,T)$ associated with the heat and momentum fluxes [see Eq. (113)],

$$\begin{aligned}\overline{\mathsf{G}}(n,T;t) &= \overline{\mathsf{F}}(n,T;0,t) - \overline{\boldsymbol{D}}(n,T;\mathbf{0},0)\overline{\boldsymbol{E}}(n,T;\mathbf{0},t)\\ &= V^{-1}\int d\Gamma\,\widetilde{\boldsymbol{f}}(\Gamma;n,T;\mathbf{0})\Bigg(\widetilde{\gamma}(\Gamma;n,T;\mathbf{0},t)\\ &\quad - \widetilde{\psi}(\Gamma;n,T;\mathbf{0})V^{-1}\int d\Gamma\,\widetilde{a}(\Gamma;n,T;\mathbf{0})\widetilde{\gamma}(\Gamma;n,T;\mathbf{0},t)\Bigg)\\ &= V^{-1}\int d\Gamma\,\widetilde{\boldsymbol{f}}(\Gamma;n,T;\mathbf{0})(1-P)\widetilde{\gamma}(\Gamma;n,T;\mathbf{0},t).\end{aligned} \tag{125}$$

Then both contributions to $\overline{\mathsf{G}}$ combine to form the projected part of the fluxes $(1-P)\tilde{\gamma}$, where P is the projection onto the set $\{\Psi_\alpha(\Gamma;n,T)\}$,

$$PX(\Gamma) = V^{-1}\Psi(\Gamma;n,T)\int d\Gamma\, a(\Gamma;n,T;\mathbf{0})X(\Gamma)$$
$$\equiv V^{-1}\Psi_\alpha(\Gamma;n,T)\int d\Gamma\, a_\alpha(\Gamma;n,T;\mathbf{0})X(\Gamma). \tag{126}$$

Thus $(1-P)$ is a projection orthogonal to the invariants, and $(1-P)\mathcal{U}(t,T)P=0$, so

$$(1-P)\mathcal{U}(t,T)\tilde{\gamma}(\Gamma;n,T) = (1-P)\mathcal{U}(t,T)(1-P)\tilde{\gamma}(\Gamma;n,T). \tag{127}$$

The time correlation function (125) then can be rewritten as

$$\overline{\mathsf{G}}_{\alpha\beta}(n,T;t) = V^{-1}\int d\Gamma\, \mathbf{\Phi}_\alpha(\Gamma;n,T)[\mathcal{U}(t,T)\mathbf{\Upsilon}(\Gamma;n,T)]_\beta$$
$$= V^{-1}\int d\Gamma\, \mathbf{\Phi}_\alpha(\Gamma;n,T)\mathcal{U}_{\beta\lambda}(t,T)\mathbf{\Upsilon}_\lambda(\Gamma;n,T) \tag{128}$$

where $\mathbf{\Phi}_\alpha$ and $\mathbf{\Upsilon}_\alpha$ are the orthogonal fluxes

$$\mathbf{\Phi}_\alpha(\Gamma;n,T) = (1-P^\dagger)\tilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\mathbf{0}),$$
$$\mathbf{\Upsilon}_\alpha(\Gamma;n,T) = (1-P)\tilde{\gamma}_\alpha(\Gamma;n,T), \tag{129}$$

with the adjoint projection operator $P^\dagger$ given by Eq. (88).

This orthogonal projection assures that there is no time-independent component of the correlation function due to the invariants. Such a contribution would not lead to a convergent limit for the time integral, as required for the transport coefficients. The property expressed by Eqs. (129) is similar to the presence of the "subtracted fluxes" in the Green-Kubo expressions for the transport coefficients of molecular fluids, with the subtracted fluxes being orthogonal to the global invariants of the dynamics [15].

VI. EULER ORDER PARAMETERS

At Euler order, the unknown parameters of the phenomenological hydrodynamics in Sec. III are the cooling rate $\zeta_0(n,T)$, the pressure $p(n,T)$, and the transport coefficient ζ^U associated with the expansion of the cooling rate to first order in the gradients. The cooling rate has been already defined by Eq. (48) above. The pressure and ζ^U can also be given explicit definitions in terms of the correlation functions from the coefficient of k in Eq. (109). Since ζ^U was previously identified as given by Eq. (111), comparison of the remaining Euler coefficients in Eqs. (17) and (109) gives

$$\begin{pmatrix} \mathcal{K}_1^{\mathrm{hyd},(a)} & 0 \\ 0 & 0 \end{pmatrix} \equiv \hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{D}}(n,T;\mathbf{0},0), \tag{130}$$

$$\mathcal{K}_1^{\mathrm{hyd},(a)} = \begin{pmatrix} 0 & 0 & n \\ 0 & 0 & \dfrac{h-e_{0,n}n}{e_{0,T}} \\ \dfrac{p_n}{nm} & \dfrac{p_T}{nm} & 0 \end{pmatrix}. \tag{131}$$

Using the definition in Eq. (97) and taking into account once again Eq. (74), it follows that

$$\hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{D}}_{\alpha\beta}(n,T;\mathbf{0},0) = V^{-1}\int d\Gamma\,\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{f}}_\alpha(\Gamma;n,T;\mathbf{0})$$
$$\times\left[\frac{\partial\rho_h(\Gamma;n,T,\mathbf{U})}{\partial y_\beta}\right]_{\boldsymbol{U}=0}. \tag{132}$$

It is shown in Appendix E that all the matrix elements of (131) follow from (132) if the pressure is identified as

$$p(n,T) \equiv (Vd)^{-1}\int d\Gamma\,\rho_h(\Gamma;n,T)\mathrm{tr}\,\mathsf{H}(\Gamma), \tag{133}$$

where $\mathrm{tr}\,\mathsf{H}(\Gamma)$ is the volume-integrated trace of the microscopic momentum flux. Its detailed form is given by Eq. (E5) of Appendix E. This is the second nontrivial result of the linear response analysis here, providing the analog of the hydrostatic pressure for a granular fluid. It is possible to show that Eq. (133) leads to $p(n,T)=nT$ in the low-density limit, but at finite density the dependence on temperature and density of the pressure is determined by details of the HCS distribution, rather than the Gibbs distribution. This is in contrast to the appearance of $e_0(n,T)$ in Eq. (130), which is a choice made in the definition of the temperature. In general, unlike in the case of normal fluids, there is no relationship of $p(n,T)$ to $e_0(n,T)$ via thermodynamics.

The transport coefficient ζ^U represents dissipation due to inelastic collisions proportional to $\boldsymbol{\nabla}\cdot\boldsymbol{U}$. It has no analog for normal fluids, where the Euler hydrodynamics is referred to as the "perfect fluid" equations, since there is no dissipation in that case. The simplest representation of ζ^U is the intermediate Helfand form, Eq. (117). More explicitly, it is shown in Appendix E that it can be expressed as

$$\zeta^U(n,T) = \lim(VTe_{0,T}d)^{-1}\int d\Gamma\, W(\Gamma;n,T)e^{-\mathcal{L}_T t}\mathcal{M}_{\zeta^U}(\Gamma;n,T), \tag{134}$$

with the source term $W(\Gamma;n,T)$ defined by

$$W(\Gamma;n,T) \equiv -LE(\Gamma) - N\left[\frac{\partial(e_{0,T}T\zeta_0)}{\partial n}\right]_{e_0}$$
$$- E(\Gamma)\left[\frac{\partial(e_{0,T}T\zeta_0)}{\partial e_0}\right]_n. \tag{135}$$

The phase function $\mathcal{M}_{\zeta^U}(\Gamma;n,T)$ is the conjugate momentum

$$\begin{aligned}\mathcal{M}_{\zeta^U}(\Gamma;n,T) &\equiv \int d\boldsymbol{r}\,\boldsymbol{r}\cdot\left[\frac{\delta\rho_{\ell h}}{\delta\boldsymbol{U}(\boldsymbol{r})}\right]_{\{y_\beta\}=\{n,T,0\}}\\ &= -\sum_{s=1}^{N}\boldsymbol{q}_s\cdot\frac{\partial\rho_h(\Gamma;n,T)}{\partial\boldsymbol{v}_s}. \qquad (136)\end{aligned}$$

The second equality makes use of the local HCS distribution form for the velocity dependence at uniform density and temperature,

$$[\rho_{lh}[\Gamma|\{y_\beta\}]]_{\{y_\beta\}=\{n,T,\boldsymbol{U}(\boldsymbol{r})\}}=\rho_h[\{\boldsymbol{q}_r,\boldsymbol{v}_r-\boldsymbol{U}(\boldsymbol{q}_r)\};n,T]. \qquad (137)$$

The corresponding Green-Kubo form follows from a similar analysis of (111), or by direct integration of Eq. (134),

$$\begin{aligned}\zeta^U(n,T)&=\lim(VTe_{0,T}d)^{-1}\int d\Gamma\; W(\Gamma;n,T)\mathcal{M}_{\zeta^U}(\Gamma;n,T)\\ &\quad-\lim(VTe_{0,T}d)^{-1}\int_0^t dt'\int d\Gamma\; W(\Gamma;n,T)\\ &\quad\times e^{-\overline{\mathcal{L}}_T t'}\overline{\mathcal{L}}_T\mathcal{M}_{\zeta^U}(\Gamma;n,T). \qquad (138)\end{aligned}$$

This completes the identification of the Euler order parameters of the linearized hydrodynamic equations, formally exact expressions for the pressure and ζ^U in terms of correlation functions for the reference HCS.

VII. NAVIER-STOKES TRANSPORT COEFFICIENTS

The six transport coefficients at order k^2 can be easily identified in terms of elements of the correlation functions matrices Λ and $\overline{Y}$ introduced in Eq. (109). The twelve intermediate Helfand and Green-Kubo forms are given in Appendix E. Only the shear viscosity is discussed here in some detail. Consider first its intermediate Helfand form. The analysis parallels closely that of ζ^U in Appendix E, with the result

$$\eta=-\lim V^{-1}\int d\Gamma\;\mathsf{H}_{xy}(\Gamma)e^{-\overline{\mathcal{L}}_T t}\mathcal{M}_\eta(\Gamma;n,T). \qquad (139)$$

Here, $\mathsf{H}_{ij}(\Gamma)$ is the volume-integrated momentum flux of Eq. (E5) in Appendix F, and $\mathcal{M}_\eta$ is the moment defined by

$$\begin{aligned}\mathcal{M}_\eta(\Gamma;n,T)&=\int d\boldsymbol{r}\,x\left[\frac{\partial\rho_{lh}}{\partial U_y(\boldsymbol{r})}\right]_{\{y_\beta\}=\{n,T,0\}}\\ &=-\sum_{r=1}^{N}q_{rx}\frac{\partial}{\partial v_{ry}}\rho_h(\Gamma;n,T). \qquad (140)\end{aligned}$$

The xy components occur here because the x axis has been taken along $\hat{\boldsymbol{k}}$ and the y axis along the transverse direction $\hat{\boldsymbol{e}}_1$, to simplify the notation. The corresponding Green-Kubo form is

$$\begin{aligned}\eta&=-\lim V^{-1}\int d\Gamma\;\mathsf{H}_{xy}(\Gamma)\mathcal{M}_\eta(\Gamma;n,T)\\ &\quad+\lim\int_0^t dt'\,V^{-1}\int d\Gamma\;\mathsf{H}_{xy}(\Gamma)e^{-\overline{\mathcal{L}}_T t'}\overline{\mathcal{L}}_T\mathcal{M}_\eta(\Gamma;n,T). \qquad (141)\end{aligned}$$

In this case, the projection operators in Eq. (128) can be omitted since their contributions vanish from symmetry, and the dynamics is orthogonal to the invariants without such terms.

The transport coefficient ζ^U vanishes for normal fluids, but the shear viscosity remains finite, as is well known. It is instructive at this point to compare and contrast the results (139) and (141) for normal and granular fluids. Suppose from the outset a local equilibrium canonical ensemble corresponding to the equilibrium $\rho_c(\Gamma)$, had been used to generate the initial perturbations. Then, assuming nonsingular conservative forces,

$$\begin{aligned}\mathcal{M}_\eta(\Gamma;n,T)&=-\sum_{r=1}^{N}q_{rx}\frac{\partial}{\partial v_{rx}}\rho_c(\Gamma)=mT^{-1}\rho_c(\Gamma)\sum_{r=1}^{N}q_{rx}v_{ry}\\ &=T^{-1}\rho_c(\Gamma)\mathsf{M}_{xy}, \qquad (142)\end{aligned}$$

with

$$\mathsf{M}_{xy}\equiv\sum_{r=1}^{N}q_{r,x}v_{r,y}. \qquad (143)$$

Then,

$$\overline{\mathcal{L}}_T\mathcal{M}_\eta(\Gamma;n,T)=T^{-1}\rho_c(\Gamma)L\mathsf{M}_{xy}=T^{-1}\rho_c(\Gamma)\mathsf{H}_{xy}(\Gamma). \qquad (144)$$

Therefore, the intermediate Helfand and Green-Kubo expressions for a normal fluid become

$$\eta=-\lim(VT)^{-1}\langle\mathsf{H}_{xy}\mathsf{M}_{xy}(-t)\rangle_c \qquad (145)$$

and

$$\eta=\lim(VT)^{-1}\int_0^t dt'\langle\mathsf{H}_{xy}\mathsf{H}_{xy}(-t')\rangle_c, \qquad (146)$$

respectively. The angular brackets denote an equilibrium canonical ensemble average, and the dynamics is that of the Liouville operator $L=\overline{L}$. These are the familiar results that have been studied and applied for more than 40 years.

To make the comparison between the elastic and inelastic cases, consider first the intermediate Helfand forms (139) and (145). The similarity between the structures for the normal and granular fluid results is striking, but the substantive changes are significant. For the granular fluid, the equilibrium ensemble has been replaced by the HCS ensemble. In addition, the Liouville operator has been replaced by that including the nonconservative force and $\overline{L}\neq L$. Finally, the generator for the dynamics includes the effect of temperature cooling in the reference HCS, $\overline{L}\rightarrow\overline{\mathcal{L}}_T=\overline{L}-\zeta_0 T\partial/\partial T$. These differences manifest themselves in the Green-Kubo expressions in analogous ways. The inclusion of nonconservative forces implies a time-independent contribution, the first term of Eq. (141), which vanishes in the elastic limit of Eq. (146). Also, due to the change in the ensemble, the two fluxes of the time correlation function differ for a granular fluid, while both are momentum fluxes for a normal fluid. Still the structure is such that these fluxes are orthogonal to the invariants of the dynamics in both cases, so that the time integrals can be expected to converge.

VIII. DIMENSIONLESS FORMS AND SCALING LIMIT

The previous section concludes the presentation of all the formal results obtained in this work. The analysis presented up to this point is quite general, and the only restriction placed on the nature of the microdynamics is that it be Markovian and the trajectories be invertible. These restrictions are satisfied by most models used to describe the interaction between granular particles. Examples are the Hertzian contact force model [28], the linear spring-dashpot model [28], and the system of inelastic hard spheres with impact-velocity-dependent coefficient of restitution [29]. The definition of these models and the associated generators of phase space dynamics are reviewed briefly in Appendix B. However, the results obtained here are in the formal language of the dynamics where both T and t are independent variables. This section shows how the formal results simplify considerably when the particles are sufficiently "hard," by introducing dimensionless variables to expose the dependence on relevant energy scales. A physically realizable limit is identified in which a scaling form for the statistical mechanics occurs allowing elimination of all reference to the temperature. This is the limit considered in more detail in the following companion paper.

In general, there are two energy scales in the problem being addressed here. One is the total energy per particle or, equivalently, the cooling temperature $T_h(t)$. The other energy scale is determined by a property of the specific collision model, called ϵ in the following. For the Hertzian spring case, it is the average compression energy of the spring. For hard spheres, it is fixed by some characteristic relative velocity in the dependence of the restitution coefficient on the relative velocity of the colliding pair. It is useful to reconsider the Liouville equation in a dimensionless form that identifies these two different scales. In the limit that their ratio $\epsilon/T_h(t)$ is small, a special scaling form of the results above is obtained. This limit holds for hard spheres with constant restitution coefficient ($\epsilon=0$), and makes precise the conditions under which that idealized model may be approximately valid for many states of interest.

Consider the Liouville equation (53) and introduce the dimensionless variables

$$\boldsymbol{q}_r^* = \frac{\boldsymbol{q}_r}{l}, \quad \boldsymbol{v}_r^* = \frac{\boldsymbol{v}_r}{v_0(T)}, \quad \epsilon^* = \frac{\epsilon}{mv_0^2(T)}, \tag{147}$$

$$s = s(t,T), \tag{148}$$

where

$$v_0(T) \equiv \left(\frac{2T}{m}\right)^{1/2} \tag{149}$$

is a thermal velocity, l is the mean free path, and the function $s(t,T)$ verifies the partial differential equation

$$\left(\frac{\partial s}{\partial t}\right)_T - \zeta_0(n,T)T\left(\frac{\partial s}{\partial T}\right)_t = \frac{v_0(T)}{l}, \tag{150}$$

with the boundary condition $s(0,T)=0$. The corresponding dimensionless distribution function is

$$\rho^*(\Gamma^*;\epsilon^*,s) = \ [lv_0(T)])^{Nd}\rho(\Gamma;n,T,t), \quad \Gamma^* \equiv \{\boldsymbol{q}_r^*,\boldsymbol{v}_r^*\}. \tag{151}$$

The dependence on the (dimensionless) density has been omitted on the left-hand side to simplify the notation. In these variables, the dimensionless Liouville equation takes the form

$$\partial_s\rho^*(\Gamma^*;\epsilon^*,s) + \bar{\mathcal{L}}^*(\Gamma^*;\epsilon^*)\rho^*(\Gamma^*;\epsilon^*,s) = 0, \tag{152}$$

where

$$\bar{\mathcal{L}}^*(\Gamma^*;\epsilon^*)\rho^* = \zeta_0^*(\epsilon^*)\epsilon^*\frac{\partial\rho^*}{\partial\epsilon^*} + \frac{\zeta_0^*(\epsilon^*)}{2}\sum_{r=1}^{N}\frac{\partial}{\partial\boldsymbol{v}_r}\cdot(\boldsymbol{v}_r^*\rho^*) + \bar{L}^*(\Gamma^*;\epsilon^*)\rho^*, \tag{153}$$

with $\bar{L}^*(\Gamma^*;\epsilon) = l\bar{L}(\Gamma)/v_0(T)$ and $\zeta_0^* = l\zeta_0(T)/v_0(T)$.

To interpret this result further, it is useful to consider the distribution of the HCS, $\rho_h^*(\Gamma^*;\epsilon^*)$, which is the steady state solution of Eq. (152), i.e.,

$$\bar{\mathcal{L}}^*(\Gamma^*;\epsilon^*)\rho_h^*(\Gamma^*;\epsilon^*) = 0. \tag{154}$$

This solution is the dimensionless form of the universal function $\rho_h(\Gamma;n,T)$, where the dependence on T has been separated into a part that simply scales the velocities, and a part that nondimensionalizes the collisional energy scale ϵ. This shows that, in the appropriate variables, the distribution function of the HCS is stationary and universal, even when velocity scaling alone (see below) does not hold. For an isolated system, $\epsilon^*(t) \equiv \epsilon/2T_h(t)$ grows with increasing t since the system temperature decreases. For very large $\epsilon^*(t)$, the collisions become elastic and the system approaches a normal fluid. However, the alternative view of (152) is to specify the solution as a function of (Γ^*,ϵ^*) and then study its properties as a special nonequilibrium steady state of granular fluids.

As just noted, for large ϵ^* the collisions become practically elastic. In the opposite limit, $\epsilon^* \ll 1$, the dependence on ϵ^* of the distribution function can be neglected,

$$\rho^*(\Gamma^*;\epsilon^*,s) \to \rho^*(\Gamma^*;s) \tag{155}$$

and the Liouville equation becomes independent of ϵ^*

$$\partial_s\rho^*(\Gamma^*) + \bar{\mathcal{L}}^*(\Gamma^*)\rho^*(\Gamma^*) = 0, \tag{156}$$

$$\overline{\mathcal{L}}^*(\Gamma^*)\rho^*(\Gamma^*) = \frac{\zeta_0^*}{2}\sum_{r=1}^{N}\frac{\partial}{\partial \boldsymbol{v}_r^*}\cdot[\boldsymbol{v}_r^*\rho^*(\Gamma^*)] + \bar{L}^*\rho^*(\Gamma^*). \tag{157}$$

This is the limit in which all temperature dependence occurs through velocity scaling alone, as there is no other significant energy scale. It occurs for sufficiently hard interactions and/or sufficiently large kinetic energy, and allows many simplifications. The HCS solution has the simple form $\rho_h^*(\Gamma^*)$ and consequently ζ_0^* is a pure number. Equation (150) defining s can be integrated in this case to give

$$s = -\frac{2}{\zeta_0^*}\ln\left(1 - \frac{\zeta_0^* v_0(T)t}{2l}\right). \tag{158}$$

Similarly, the dimensionless form of the cooling equation for $T_h(t)$, Eq. (45), can be integrated to get the explicit dependence on t

$$\frac{T_h(t)}{T_h(0)} = \left(1 + \frac{\zeta_0^* v_0[T_h(0)]t}{2l}\right)^{-2}. \tag{159}$$

Finally, when (157) is evaluated at $T_h(t)$ the relationship of s to t becomes

$$s = \frac{2}{\zeta_0^*}\ln\left(1 + \frac{v_0(0)}{2l}\zeta_0^* t\right). \tag{160}$$

The conditions for which Eq. (155) applies will be called the "scaling limit."

The special collisional model of inelastic hard spheres with constant restitution coefficient has no intrinsic collisional energy scale, so $\boldsymbol{\epsilon}^*=0$ and the scaling limit applies. The generators for this dynamics are indicated in Appendix B and can be understood as the singular limit of a soft, continuous potential, like the Hertzian contact force model. The analysis of the preceding sections is specialized to this case in the following paper [18], where it is shown that the above simplifications admit a more detailed exposition of the formal expressions for the transport coefficients. The rest of this section is a brief translation of some of the main results here to their dimensionless, scaling limit form.

The dimensionless hydrodynamic fields are defined by

$$\{\delta y_\alpha^*\} \equiv \left\{\frac{\delta y_\alpha}{\bar{y}_{\alpha,h}}\right\} \equiv \left\{\frac{\delta n}{n_h}, \frac{\delta T}{T_h}, \frac{\delta \boldsymbol{U}}{v_0(T)}\right\}, \tag{161}$$

where the definition of the $\bar{y}_{\alpha,h}$'s follows from the second identity above. Then, the fundamental linear response equation Eq. (68) in the dimensionless variables $\delta\widetilde{y}_\alpha^*$ is

$$\delta\widetilde{y}^*(\boldsymbol{k}^*,s) = \widetilde{C}^*(\boldsymbol{k}^*,s)\,\delta\widetilde{y}^*(\boldsymbol{k}^*,0), \tag{162}$$

with

$$\widetilde{C}_{\alpha\beta}^*(\boldsymbol{k}^*,s) = \frac{1}{\bar{y}_{\alpha,h}[T_h(t)]}\widetilde{C}_{\alpha\beta}[n_h,T_h(t);\boldsymbol{k},t]\bar{y}_{\beta,h}[T_h(0)]. \tag{163}$$

It follows from Eq. (72) that $\widetilde{C}_{\alpha\beta}^*(\boldsymbol{k}^*,s)$ obeys the equation

$$[\partial_s + \mathcal{K}^*(\boldsymbol{k}^*,s)]\widetilde{C}^*(\boldsymbol{k}^*,s) = 0, \quad \widetilde{C}^*(\boldsymbol{k}^*,0) = I, \tag{164}$$

the dimensionless transport matrix being

$$\mathcal{K}_{\alpha\beta}^*(\boldsymbol{k}^*,s) = -\,\delta_{\alpha\beta}p_\alpha\zeta_0^* + \frac{l\bar{y}_{\beta,h}[T(t)]}{v_0[T_h(t)]\bar{y}_{\alpha,h}[T(t)]}\mathcal{K}_{\alpha\beta}(n,T;\boldsymbol{k},t). \tag{165}$$

Here $\mathcal{K}(n,T;\boldsymbol{k},t)$ is the transport matrix analyzed in the previous sections. The additional contributions to $\mathcal{K}^*(\boldsymbol{k}^*,s)$, proportional to $\{p_\alpha\}\equiv\{0,1,\frac{1}{2},\ldots,\frac{1}{2}\}$, arise from differentiating the normalization constants with respect to T. Because of the scaling limit, $\mathcal{K}_{\alpha\beta}^*(\boldsymbol{k}^*,s)$ is independent of $T(t)$ and the hydrodynamic limit can be identified as

$$\mathcal{K}^{*\text{hyd}}(\boldsymbol{k}^*) = \mathcal{K}^*(\boldsymbol{k}^*,\infty). \tag{166}$$

The phenomenological form of $\mathcal{K}^{*\text{hyd}}(\boldsymbol{k}^*)$, corresponding to that of Sec. III above, is given in the following companion paper [18].

The dimensionless forms for the response functions $\widetilde{C}_{\alpha\beta}^*(\boldsymbol{k}^*,s)$ as phase space averages follow from Eq. (70),

$$\begin{aligned}\widetilde{C}_{\alpha\beta}^*(\boldsymbol{k}^*,s) &= \left[\frac{1}{V\bar{y}_{\alpha,h}(T)}\int d\Gamma\ \widetilde{a}_\alpha(\Gamma;n,T,\boldsymbol{k})\right.\\ &\quad\left.\times e^{-t\mathcal{L}_T}\widetilde{\psi}_\beta(\Gamma;n,T,-\boldsymbol{k})\right]_{n=n_h,T=T_h(t)}\bar{y}_{\beta,h}[T(0)]\\ &= \frac{1}{Vy_{\alpha,h}[T_h(t)]}\int d\Gamma\ \widetilde{a}_\alpha[\Gamma;n_h,T_h(t);\boldsymbol{k}]\\ &\quad\times e^{-t\bar{L}}\widetilde{\psi}_\beta[\Gamma;n_h,T_h(0);-\boldsymbol{k}]\bar{y}_{\beta,h}[T_h(0)]\\ &= V^{*-1}\int d\Gamma^*\widetilde{a}_\alpha^*(\Gamma^*;\boldsymbol{k}^*)e^{-s\overline{\mathcal{L}}^*}\widetilde{\psi}_\beta^*(\Gamma^*;-\boldsymbol{k}^*),\end{aligned} \tag{167}$$

where $V^*=V/l^d$. More details of this transformation are given in Appendix B of Ref. [18]. The generator for the dynamics $\overline{\mathcal{L}}^*$ is given by Eq. (156), and the dimensionless phase functions are

$$\widetilde{a}_\alpha^*(\Gamma^*;\boldsymbol{k}^*) = \frac{\widetilde{a}_\alpha(\Gamma;n,T,\boldsymbol{k})}{l^d\bar{y}_{\alpha,h}(T)}, \tag{168}$$

$$\widetilde{\psi}_\beta^*(\Gamma^*;\boldsymbol{k}^*) = [v_0(T)l]^{Nd}\widetilde{\psi}_\beta(\Gamma;n,T;\boldsymbol{k})\bar{y}_{\beta,h}(T). \tag{169}$$

The hydrodynamic transport matrix $\mathcal{K}^{*\text{hyd}}(\boldsymbol{k}^*)$ is therefore given by

$$\begin{aligned}\mathcal{K}^{*\text{hyd}}(\boldsymbol{k}^*) &= \mathcal{K}^{*\text{hyd}}(\boldsymbol{0}) - \lim\{[\partial_s + \mathcal{K}^{*\text{hyd}}(\boldsymbol{0})]\widetilde{C}^*(\boldsymbol{k}^*,s)\}\\ &\quad\times\widetilde{C}^{*-1}(\boldsymbol{k}^*,s),\end{aligned} \tag{170}$$

which is the dimensionless form of Eq. (73). In this way, all relation to the cooling of the reference state through a dependence on T has been removed. However, the dynamics of homogeneous perturbations of this state remains through

$$\widetilde{C}^*(\boldsymbol{0},s) = e^{-\mathcal{K}^{*\text{hyd}}(\boldsymbol{0})s}. \tag{171}$$

The dimensionless correlation functions defining the transport coefficients are obtained in a similar way and have representations analogous to (167). An important difference is that the generator is $\overline{\mathcal{L}}^{*}-\mathcal{K}^{\text{hyd}*}(\mathbf{0})$, indicating that the homogeneous hydrodynamics is compensated. These simplifications and further interpretation are also deferred to the following paper.

IX. DISCUSSION

The objective of this work has been to translate the familiar methods of linear response for normal fluids to the related, but quite different case, of granular fluids. In both cases, the linear response to perturbations of a homogeneous reference state is described in terms of the fundamental tools of nonequilibrium statistical mechanics. This microscopic formulation is then compared with the corresponding description from phenomenological hydrodynamics, and the unknown parameters of the latter are identified in terms of associated reference state response functions. The analysis entails several steps, and at each stage there are technical and conceptual differences encountered for granular fluids that have been addressed in the preceding sections. Given the complicated technical nature of this work, this final section provides a brief summary of the new features for granular fluids and a summary of the primary results of this work. The context and utility of these results is also addressed.

Generalizing linear response methods to granular fluids entails several important differences from normal fluids. Primary among these are the following.

(1) The homogeneous reference state about which perturbations are considered is not the Gibbs state, but the HCS state. It is not simply a function of the global invariants of number, energy, and momentum, but must be determined as the self-consistent solution to Eqs. (48) and (49).

(2) The reference state is time dependent, due to the collisional energy loss. Here, it has been given a stationary representation by including the temperature as a dynamical variable [see Eqs. (50) and (53)]. In this form, the granular linear response problem becomes the response to the spatial perturbation of a homogeneous, stationary state, similar to that for a normal fluid. However, the generators for that dynamics are now more complex, due both to the nonconservative forces responsible for collisional energy loss and a generator for changes in the granular temperature.

(3) The response functions contain information about both hydrodynamic and microscopic collective excitations. Extraction of the hydrodynamic modes at long wavelengths and long times can lead to complex formal expressions of little use. However, if the perturbation is chosen to excite only the hydrodynamic modes this analysis becomes simpler and more direct. Practically, this can be done only in the long-wavelength limit. For a normal fluid, the long-wavelength hydrodynamic modes are the dynamical invariants and the associated perturbation is generated by the corresponding densities of global conserved quantities (number, energy, momentum). The granular fluid is more complex, since there is a nontrivial hydrodynamics even in the extreme long-wavelength limit, due to the homogeneous *nonlinear* temperature cooling that is linearized about a reference cooling state. The hydrodynamic modes are therefore identified from this nonzero long-wavelength dynamics, and identification of the perturbations corresponding to this dynamics is the analog of finding the invariants for a normal fluid. This has been addressed in Sec. IV. When the homogeneous dynamics of cooling and its homogeneous response are included in the Liouville dynamics, these perturbations become the new invariants for a granular fluid [see Eqs. (61) and (62)].

(4) With this knowledge of the special long-wavelength hydrodynamic perturbations, spatial perturbations are constructed from their corresponding local densities, just as for a normal fluid. In the latter case, these are the microscopic local densities of number, energy, and momentum, and are generated by a local equilibrium ensemble. In the same way, the densities of the invariants for the granular fluid are generated from a corresponding local HCS. Since the HCS is not the Gibbs state, these densities are no longer the local conserved densities for a normal fluid, but instead are new conserved quantities associated with the generator of dynamics for spatial perturbations.

Resolution of these important conceptual issues has led to the identification of formally exact expressions for all of the phenomenological parameters of Navier-Stokes hydrodynamics in terms of HCS averages and the dynamics of HCS fluctuations. This is the formal objective of linear response for both normal and granular fluids—representation of spatial excitations in terms of the reference homogeneous state. Such results provide the appropriate basis for further practical studies of how these parameters depend in detail on the state conditions, such as density and temperature. That is the task of many-body theory or molecular dynamics simulations.

To summarize, the primary results obtained here are as follows.

(1) The cooling rate and the pressure in the linear hydrodynamic equations are identified as specific averages over the HCS solution. In particular, the pressure is the same average of the trace of the microscopic stress tensor as for an equilibrium fluid, but with the equilibrium Gibbs distribution replaced by the HCS. This shows how the dependence of the pressure on density and temperature is to be determined.

(2) The transport coefficients are of two types, those associated with the heat and momentum fluxes, and those associated with the cooling rate. In each case they can be displayed collectively in matrix form. Those associated with the fluxes have a Green-Kubo representation,

$$\Lambda_{\alpha\beta}(n,T)=\Lambda_{\alpha\beta}^{(0)}(n,T)-\lim\int_{0}^{t}dt'\,G_{\alpha\beta}(n,T;t'). \quad (172)$$

The first term on the right-hand side is a time-independent correlation function for the HCS. It vanishes for normal fluids with continuous potentials of interaction, and occurs here due to the inelasticity of the collisions. Such a term can occur even in the elastic limit for singular forces, such as hard spheres [30]. The limit indicated in the second term is the usual thermodynamic limit of large volume V and particle number N, followed by the limit of large time. The integrand $G_{\alpha\beta}(n,T;t)$ is a flux-flux correlation function

$$G_{\alpha\beta}(n,T;t) \equiv \frac{1}{V}\int d\Gamma\ \Phi_{\alpha}(\Gamma;n,T)\mathcal{U}_{\beta\lambda}(t,T)\Upsilon_{\lambda}(\Gamma;n,T). \tag{173}$$

The first flux Φ_{α} is one of those associated with the usual densities of number, energy, and momentum. The second flux Υ_{λ} is one of those associated with the densities for the new granular fluid invariants. The evolution operator for the dynamics $\mathcal{U}_{\alpha\beta}(t,T)$ is the usual Liouville dynamics, but in addition compensated for the homogeneous dynamics of collisional cooling and its homogeneous perturbation. This shifts the spectrum of the generator for the dynamics so that the homogeneous dynamics becomes stationary with corresponding invariants. These would pose a problem for the limit in (172) if it were not for the fact that the fluxes $\Upsilon_{\lambda}(\Gamma;n,T)$ are all orthogonal to these invariants. This is similar to the Green-Kubo expressions for a normal fluid, and is an important self-consistency requirement for the analysis.

Finally, the context and utility of the results in this work are discussed. The analysis has focused on the response of the system for small perturbations from the HCS. In many experimental conditions of interest, for both normal and granular fluids, the system is not close to a global homogeneous state. Nevertheless, the reference states studied here are relevant *locally* for more complex and realistic physical conditions [31]. For example, the transport coefficients such as viscosity and thermal conductivity obtained from linear response, are the same functions of density and temperature as those in the associated nonlinear equations applicable under more general conditions. Thus the context of relevance of these results transcend the limitations associated with the starting point of the linear response analysis itself and extend to states for which the nonlinear Navier-Stokes equations are required.

Stated differently, the transport coefficients do not have the limitations of approximate solutions to the hydrodynamic equations in which they occur. For example, the linearized hydrodynamic equations considered here are valid only over an initial time interval due to a long-wavelength instability, after which the full nonlinear equations must be used. The transport coefficients characterize that instability, but do not depend on it. This can be seen from the form of (172) which is strictly translationally invariant and whose homogeneous hydrodynamics has been projected out. The long-time limit can be taken even though its expression has been extracted from response functions applicable for only a finite time. These same expressions for the transport coefficients have been rederived recently by a different method leading to the nonlinear hydrodynamic equations, for which the instability is not an issue [32].

The utility of these formal results rests on further studies of appropriate ways to evaluate them. The status now is that expressions for the quantities of interest (e.g., pressure and transport coefficients) are given formally without any inherent uncontrolled approximations (e.g., as in some chosen kinetic theory) in terms of the reference HCS. This is the appropriate point for the introduction of practical methods for evaluation, and is analogous to the study of normal fluids where the pressure and transport coefficients are given only formally in terms of the reference Gibbs state. A number of methods have proved useful, including molecular dynamics simulations, density expansions, memory function models incorporating initial dynamics, and linear kinetic theory. The formal expressions obtained here provide the basis to explore the relevance of these and similar approaches for granular gases. As an example, consider again the new Euler order transport coefficient given by Eq. (134), make the temperature scaling of the generator explicit, and evaluate the entire expression at $T=T_h(t)$ to get the result

$$\zeta^{U}[n,T_h(t)] = -\lim[VT_h(t)e_{0,T}[T_h(t)]d]^{-1} \times\int d\Gamma\ W[\Gamma;n,T_h(t)]e^{-\bar{L}t}\sum_{s=1}^{N}\boldsymbol{q}_s\cdot\frac{\partial\rho[\Gamma;n,T(0)]}{\partial\boldsymbol{v}_s}. \tag{174}$$

The generator for the dynamics is now that for the trajectories alone, so this is a form suitable for molecular dynamics simulation. The simulation method must be constructed in such a way as to represent the HCS appearing in (174). Elsewhere, the evaluation by kinetic theory [13] is described, and the original representation given by Eq. (134) is found to be more suitable.

It is worth recalling that liquid state transport for simple atomic fluids remains a prototypical strongly coupled many-body problem, with limited progress beyond simulation of formal expressions such as those given here. More should not be expected for "complex" granular fluids. The formal representations of transport coefficients by methods of statistical mechanics provides a new perspective on a difficult old problem. As for normal fluids, significant further progress can be expected for the idealized model of hard spheres. That is the subject of the following companion paper.

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APPENDIX A: HOMOGENEOUS STATE DYNAMICS

The dynamics associated with the homogeneous cooling state of interest here is twofold. The first is the cooling of the temperature, determined from the solution to Eq. (12). For a given initial condition T, the solution is denoted by $T_h(t;n_h,T)$. The density is a constant parameter and sometimes it is left implicit in the notation of the text. The second dynamics is the linear response to small homogeneous changes in the initial conditions,

$$\delta T_h(t;n_h,T) = \left(\frac{\partial T_h(t;n_h,T)}{\partial n_h}\right)_T \delta n_h + \left(\frac{\partial T_h(t;n_h,T)}{\partial T}\right)_{n_h} \delta T. \tag{A1}$$

In this appendix, it is shown how the response function for this second type of dynamics is obtained from the linearized hydrodynamic equations to give Eq. (31).

A useful identity for any function of the temperature, $X(T)$ is

$$X[T(t_2)] = \exp\left(-(t_2-t_1)\zeta_0[T(t_1)]T(t_1)\frac{\partial}{\partial T(t_1)}\right)X[T(t_1)]. \tag{A2}$$

Here $T(t)$ is a solution to Eq. (12). The identity can be proved by performing a Taylor series of $X[T(t_2)]$ in powers of (t_2-t_1) and using Eq. (12) to evaluate the time derivatives in terms of $T(t_1)$ derivatives. The above identity gives, in particular,

$$T(0) = \exp\left(t\zeta_0[T(t)]T(t)\frac{\partial}{\partial T(t)}\right)T(t),$$

$$T(t) = \exp\left(-t\zeta_0[T(0)]T(0)\frac{\partial}{\partial T(0)}\right)T(0), \tag{A3}$$

and

$$\left(\frac{\partial T(t)}{\partial T(0)}\right)_{n_h} = \frac{\zeta_0[T(t)]T(t)}{\zeta_0[T(0)]T(0)}. \tag{A4}$$

Consider some function $X[T(t;T),t]$ that depends on time through $T(t;T)$ plus some residual time dependence. Use the second equation of (A3) to write

$$X[T(t;T),t] = \exp[-t\zeta_0(T)T\partial_T]X(T,t) \tag{A5}$$

and, consequently,

$$\begin{aligned}\partial_t X[T(t;T),t] &= \exp[-t\zeta_0(T)T\partial_T]\{[\partial_t - \zeta_0(T)T\partial_T]X(T,t)\}\\ &= \{[\partial_t - \zeta_0(T)T\partial_T]X(T,t)\}_{T=T(t;T)}.\end{aligned} \tag{A6}$$

The time dependence due to $T(t;T)$ can be replaced by treating T as an independent variable, with the additional generator for its dynamics $\zeta_0 T\partial_T$. It is then equivalent to determine $X(T,t)$, and evaluate it finally at $T=T(t;T)$. In the case of Eq. (23), this leads to

$$[\partial_t - \zeta_0(T)T\partial_T + \mathcal{K}^{\text{hyd}}(n,T;\boldsymbol{k})]\tilde{C}^{\text{hyd}}(n,T;\boldsymbol{k},t) = 0, \tag{A7}$$

$$\tilde{C}^{\text{hyd}}_{\alpha\beta}(n,T;\boldsymbol{k},0) = \delta_{\alpha\beta}, \tag{A8}$$

with the definition in Eq. (26). For $\boldsymbol{k}=\boldsymbol{0}$, use of Eqs. (17)–(19) gives

$$\tilde{C}^{\text{hyd}}_{\alpha\beta}(n,T;\boldsymbol{0},t) = \delta_{\alpha\beta} \tag{A9}$$

for $\alpha\neq 2$, while for $\alpha=2$ we get:

$$\begin{aligned}&[\partial_t - \zeta_0(T)T\partial_T + \mathcal{K}^{\text{hyd}}_{22}(n,T;\boldsymbol{0})]\tilde{C}^{\text{hyd}}_{2\beta}(n,T;\boldsymbol{0},t)\\ &\quad + \mathcal{K}^{\text{hyd}}_{21}(n,T;\boldsymbol{0})\,\delta_{1\beta} = 0,\end{aligned} \tag{A10}$$

or, more explicitly,

$$\left[\partial_t - \zeta_0 T\partial_T + \left(\frac{\partial(\zeta_0 T)}{\partial T}\right)_n\right]\tilde{C}^{\text{hyd}}_{2\beta}(n,T;\boldsymbol{0},t) + \left(\frac{\partial(\zeta_0 T)}{\partial n}\right)_T \delta_{1\beta} = 0. \tag{A11}$$

The solution of this equation is

$$\tilde{C}^{\text{hyd}}_{2\beta}(n,T;\boldsymbol{0},t) = \left(\frac{\partial T}{\partial n}\right)_{T(-t;T)}\delta_{1\beta} + \left(\frac{\partial T}{\partial T(-t;T)}\right)_n \delta_{2\beta}, \tag{A12}$$

as can be verified by direct substitution into Eq. (A11) and repeated use of Eq. (A4). This is the result (31) of the text.

APPENDIX B: GENERATORS OF DYNAMICS

The interaction between the constituent particles of the dissipative fluid enters the presentation here via the Liouville operators that generate the dynamics. The analysis of the text places few restrictions on these generators and admits a large class of models to represent real systems. For example, it is not necessary that they be pairwise additive, although the examples of this appendix all assume that case. There is a qualitative difference between the generators for continuous or piecewise continuous forces, and those for singular forces (e.g., hard spheres). Examples of each are given here for illustration.

1. Dissipative soft spheres

The fluid is assumed to be comprised of monodisperse spherical particles with pairwise additive central interactions. The latter implies that the forces are "smooth," without tangential momentum transfer, and Newton's third law holds. The simplest realistic model for the force $\boldsymbol{F}$ that particle s exerts on particle r is the smooth, frictional contact model [28,29] given by

$$F(\boldsymbol{q}_{rs},\boldsymbol{g}_{rs}) = \hat{\boldsymbol{q}}_{rs}\Theta(\sigma-q_{rs})[f(\sigma-q_{rs}) - \gamma(\sigma-q_{rs})(\boldsymbol{g}_{rs}\cdot\hat{\boldsymbol{q}}_{rs})], \tag{B1}$$

where $\Theta(x)$ is the Heaviside step function, $\boldsymbol{q}_{rs}\equiv\boldsymbol{q}_r-\boldsymbol{q}_s$ is the relative coordinate, $\boldsymbol{g}_{rs}\equiv\boldsymbol{v}_r-\boldsymbol{v}_s$ is the relative velocity of the two particles, and $\hat{\boldsymbol{q}}_{rs}\equiv\boldsymbol{q}_{rs}/q_{rs}$ is the unit normal vector joining the centers of the two particles. Moreover, $f(x)$ and γ are a function and a constant, respectively, to be described below. This is a piecewise continuous force that vanishes for separations greater than σ, which therefore can be thought of as the diameter of the particles. The first term between the square brackets describes a *conservative* force representing the elastic repulsion due to the deformation of real granular particles. If $f(x)$ is chosen to be linear, the deformation is that of a spring. The amount of deformation can be adjusted by the choice of the spring constant. A second, more realistic, choice is the Hertzian contact model for which $f(x)\propto x^{3/2}$.

The second term of (B1) is a *nonconservative* force representing the energy loss of the particle pair on collision. It is proportional to the relative velocity of approach during the collision, and the amount of energy loss is adjusted by the choice of the friction constant γ.

The Liouville operators L and $\bar{L}$, defined in Eqs. (35) and (36), for the dynamics of phase functions and distributions for these models can be identified as

$$LX(\Gamma) \equiv \sum_{r=1}^{N} \boldsymbol{v}_r \cdot \frac{\partial}{\partial \boldsymbol{q}_r} X(\Gamma) + \frac{1}{m} \sum_{r=1}^{N} \sum_{s \neq r}^{N} \boldsymbol{F}(\boldsymbol{q}_{rs}, \boldsymbol{g}_{rs}) \cdot \frac{\partial}{\partial \boldsymbol{v}_r} X(\Gamma) \tag{B2}$$

and

$$\bar{L}X(\Gamma) \equiv LX(\Gamma) + \frac{1}{m} \sum_{r=1}^{N} \sum_{r \neq s}^{N} X(\Gamma) \frac{\partial}{\partial \boldsymbol{v}_r} \cdot \boldsymbol{F}(\boldsymbol{q}_{rs}, \boldsymbol{g}_{rs}). \tag{B3}$$

It is readily verified that the total momentum is conserved, since Newton's third law is satisfied, i.e., $\boldsymbol{F}(\boldsymbol{q}_{rs}, \boldsymbol{g}_{rs}) = -\boldsymbol{F}(\boldsymbol{q}_{sr}, \boldsymbol{g}_{sr})$. The total energy is

$$E(\Gamma) = \sum_{r=1}^{N} \frac{1}{2} m v_r^2 + \frac{1}{2} \sum_{r=1}^{N} \sum_{s \neq r}^{N} V(q_{rs}), \tag{B4}$$

where the potential energy function $V(q)$ satisfies

$$\frac{\partial V(q_{rs})}{\partial q_{rs}} = -\Theta(\sigma - q_{rs}) f(\sigma - q_{rs}). \tag{B5}$$

The microscopic energy loss is easily computed by using Eq. (38) with the result

$$LE(\Gamma) = -\frac{1}{2} \sum_{r=1}^{N} \sum_{s \neq r}^{N} \Theta(\sigma - q_{rs}) \gamma(\sigma - q_{rs}) (\boldsymbol{g}_{rs} \cdot \hat{\boldsymbol{q}}_{rs})^2, \tag{B6}$$

showing that it is associated with the nonconservative part of the interactions as it should be.

2. Hard sphere dynamics

For a given energy of activation, the contact forces considered above may have small deformations, i.e., the region in which the forces differs from zero verifies $(\sigma - q_{rs})/\sigma \ll 1$. In that case, the conservative part of the force approaches that of elastic hard spheres. The primary effect of the nonconservative force is to decrease the magnitude of $\boldsymbol{g}_{rs} \cdot \hat{\boldsymbol{q}}_{rs}$ after the collision. This can be represented by the scattering law

$$\boldsymbol{g}_{rs}' = \boldsymbol{g}_{rs} - [1 + \alpha(g_{rs})](\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{g}_{rs}) \hat{\boldsymbol{\sigma}}, \tag{B7}$$

where $\boldsymbol{g}_{rs}'$ is the relative velocity after collision and $\alpha(g_{rs})$ is a coefficient of restitution that depends on the relative velocity. The total momentum of the pair is, by definition, unchanged in the collision. The elastic limit corresponds to $\alpha(g_{rs}) \to 1$. Subsequent to the change in relative velocity for the pair (r,s), the free streaming of all particles continues until another pair is at contact, and the corresponding instantaneous change in their relative velocities is performed. The collision rule is assumed to be invertible, i.e., $\alpha(g_{rs})$ is specified so that the trajectory can be reversed.

Since there is no longer a potential energy, the total energy for the system is its kinetic energy, which changes on a pair collision by

$$\Delta\left(\frac{1}{2} m (v_r^2 + v_s^2)\right) = \frac{1}{4} m (g_{rs}'^2 - g_{rs}^2)$$

$$= -\frac{1}{4} m [1 - \alpha^2(g_{rs})](\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{g}_{rs})^2. \tag{B8}$$

This is clearly the analog of the terms on the right-hand side of Eq. (B14). In fact, the velocity dependence of the coefficient of restitution can be modeled from a comparison of the two equations.

There are two components of the generators L and $\bar{L}$, corresponding to each of the two steps of free streaming and velocity changes at contact,

$$L = \sum_{r=1}^{N} \boldsymbol{v}_r \cdot \frac{\partial}{\partial \boldsymbol{r}_r} + \frac{1}{2} \sum_{r=1}^{N} \sum_{s \neq r}^{N} T(r,s), \tag{B9}$$

$$\bar{L} = \sum_{r=1}^{N} \boldsymbol{v}_r \cdot \frac{\partial}{\partial \boldsymbol{r}_r} - \frac{1}{2} \sum_{r=1}^{N} \sum_{s \neq r}^{N} \bar{T}(r,s). \tag{B10}$$

The operators $T(r,s)$ and $\bar{T}(r,s)$ describe the binary collision for a pair,

$$T(r,s) = \delta(q_{rs} - \sigma) \Theta(-\boldsymbol{g}_{rs} \cdot \hat{\boldsymbol{q}}_{rs}) |\boldsymbol{g}_{rs} \cdot \hat{\boldsymbol{q}}_{rs}| (b_{rs} - 1), \tag{B11}$$

$$\bar{T}(r,s) = \delta(q_{rs} - \sigma)[J(\boldsymbol{v}_r, \boldsymbol{v}_s) b_{rs}^{-1} - 1] \Theta(-\boldsymbol{g}_{rs} \cdot \hat{\boldsymbol{q}}_{rs}) |\boldsymbol{g}_{rs} \cdot \hat{\boldsymbol{q}}_{rs}|. \tag{B12}$$

Here b_{rs} is a substitution operator,

$$b_{rs} X(\boldsymbol{g}_{rs}) = X(b_{rs} \boldsymbol{g}_{rs}) = X(\boldsymbol{g}_{rs}'), \tag{B13}$$

which changes the relative velocity $\boldsymbol{g}_{rs}$ into its scattered value $\boldsymbol{g}_{rs}'$, and b_{rs}^{-1} is its inverse. Finally, $J(\boldsymbol{v}_r, \boldsymbol{v}_s)$ is the Jacobian for the transformation from $\{\boldsymbol{v}_r, \boldsymbol{v}_s\}$ to $\{\boldsymbol{v}_r', \boldsymbol{v}_s'\}$,

$$J(\boldsymbol{v}_r, \boldsymbol{v}_s) = \left| \frac{\partial(b_{rs} \boldsymbol{v}_r, b_{rs} \boldsymbol{v}_s)}{\partial(\boldsymbol{v}_r, \boldsymbol{v}_s)} \right|^{-1}. \tag{B14}$$

The δ function in (B11) and (B12) requires that the pair is at contact, while the Θ function requires that the directions of velocities are such as to assure a collision. A derivation of these results and further details are given in the companion paper following this one.

APPENDIX C: HOMOGENEOUS COOLING SOLUTION

In this appendix, Eqs. (55) and (57) are proved, leading to the solution of the Liouville Eq. (59) for homogeneous perturbations of the HCS. The HCS is the stationary solution to the Liouville equation (52), i.e.,

$$\overline{\mathcal{L}}_T\rho_h(\Gamma;n,T,\boldsymbol{U})=0, \tag{C1}$$

$$\overline{\mathcal{L}}_T\equiv-\zeta_0(n,T)T\partial_T+\bar{L}. \tag{C2}$$

The action of the operator $\overline{\mathcal{L}}_T$ on Ψ_α can be evaluated as follows:

$$\begin{aligned}\overline{\mathcal{L}}_T\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})&=\left(\frac{\partial[\overline{\mathcal{L}}_T\rho_h(\Gamma;n,T,\boldsymbol{U})]}{\partial y_\alpha}\right)_{y_{\beta\neq\alpha}}\\&\quad+\left(\frac{\partial[\zeta_0(n,T)T]}{\partial y_\alpha}\right)_{y_{\beta\neq\alpha}}\frac{\partial\rho_h(\Gamma;n,T,\boldsymbol{U})}{\partial T}\\&=\left(\frac{\partial[\zeta_0(n,T)T]}{\partial y_\alpha}\right)_{y_{\beta\neq\alpha}}\Psi_2(\Gamma;n,T,\boldsymbol{U})\\&=\Psi_\beta(\Gamma;n,T,\boldsymbol{U})\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0}),\end{aligned} \tag{C3}$$

where $\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0})$ has been identified in the last equality from Eqs. (17)–(19) particularized for $\boldsymbol{k}=\boldsymbol{0}$. This proves Eq. (55).

Next, using this result,

$$\begin{aligned}\overline{\mathcal{L}}_T^2\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})&=[\overline{\mathcal{L}}_T\Psi_\beta(\Gamma;n,T,\boldsymbol{U})]\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0})\\&\quad+\Psi_\beta(\Gamma;n,T,\boldsymbol{U})\overline{\mathcal{L}}_T\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0})\\&=\Psi_\gamma(\Gamma;n,T,\boldsymbol{U})\mathcal{K}^{\text{hyd}}_{\gamma\beta}(n,T;\boldsymbol{0})\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0})\\&\quad+\Psi_\beta(\Gamma;n,T,\boldsymbol{U})\\&\quad\times[-\zeta_0(n,T)T\partial_T]\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0})\\&=\Psi_\gamma(\Gamma;n,T,\boldsymbol{U})[\mathcal{K}^{\text{hyd}}_{\gamma\beta}(n,T;\boldsymbol{0})\\&\quad-\delta_{\gamma\beta}\zeta_0(n,T)T\partial_T]\mathcal{K}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0}),\end{aligned} \tag{C4}$$

or, in a compact matrix notation,

$$\overline{\mathcal{L}}_T^2\Psi=\Psi(\mathcal{K}^{\text{hyd}}-I\zeta_0T\partial_T)\mathcal{K}^{\text{hyd}}, \tag{C5}$$

with I denoting here the $d+2$ unit matrix. Applying the induction method,

$$\begin{aligned}\overline{\mathcal{L}}_T^l\Psi&=\overline{\mathcal{L}}_T[\Psi(\mathcal{K}^{\text{hyd}}-I\zeta_0T\partial_T)^{l-2}\mathcal{K}^{\text{hyd}}]\\&=(\overline{\mathcal{L}}_T\Psi)(\mathcal{K}^{\text{hyd}}-I\zeta_0T\partial_T)^{l-2}\mathcal{K}^{\text{hyd}}\\&\quad+\Psi_T[(\mathcal{K}^{\text{hyd}}-I\zeta_0T\partial_T)^{l-2}\mathcal{K}^{\text{hyd}}]\\&=(\Psi\mathcal{K}^{\text{hyd}})(\mathcal{K}^{\text{hyd}}-I\zeta_0T\partial_T)^{l-2}\mathcal{K}^{\text{hyd}}\\&\quad+\Psi(-\zeta_0T\partial_T)(\mathcal{K}^{\text{hyd}}-I\zeta_0T\partial_T)^{l-2}\mathcal{K}^{\text{hyd}}\\&=\Psi(\mathcal{K}^{\text{hyd}}-I\zeta_0T\partial_T)^{l-1}\mathcal{K}^{\text{hyd}}\\&=\Psi(\mathcal{K}^{\text{hyd}}-I\zeta_0T\partial_T)^{l}.\end{aligned} \tag{C6}$$

This implies

$$\begin{aligned}e^{-\overline{\mathcal{L}}_Tt}\Psi_\alpha(\Gamma;n,T,\boldsymbol{U})&=\Psi_\beta(\Gamma;n,T,\boldsymbol{U})\{\exp-[\mathcal{K}^{\text{hyd}}(n,T;\boldsymbol{0})\\&\quad-I\zeta_0(n,T)T\partial_T]t\}_{\beta\alpha}\\&=\Psi_\beta(\Gamma;n,T,\boldsymbol{U})\widetilde{C}^{\text{hyd}}_{\beta\alpha}(n,T;\boldsymbol{0},t),\end{aligned} \tag{C7}$$

which proves Eq. (57). In the last transformation, the formal solution of Eq. (27) has been used.

APPENDIX D: MICROSCOPIC CONSERVATION LAWS (BALANCE EQUATIONS)

1. Fluxes associated with $\tilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k})$

The microscopic balance equations for the phase functions $\tilde{a}_\alpha(\Gamma;n,T;\boldsymbol{k})$ follow from those for the Fourier-transformed number density $\widetilde{\mathcal{N}}(\Gamma;\boldsymbol{k})$, energy density $\widetilde{\mathcal{E}}(\Gamma;\boldsymbol{k})$, and momentum density $\widetilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{k})$ defined in Eq. (65). These balance equations relate the time dependence of the densities to appropriate fluxes

$$\partial_te^{Lt}\begin{pmatrix}\widetilde{\mathcal{N}}(\Gamma;\boldsymbol{k})\\\widetilde{\mathcal{E}}(\Gamma;\boldsymbol{k})\\\widetilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{k})\end{pmatrix}=i\boldsymbol{k}\cdot e^{Lt}\begin{pmatrix}\dfrac{\widetilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{k})}{m}\\\tilde{\boldsymbol{s}}(\Gamma;\boldsymbol{k})\\\tilde{\mathsf{h}}(\Gamma;\boldsymbol{k})\end{pmatrix}-e^{Lt}\begin{pmatrix}0\\\tilde{w}(\Gamma;\boldsymbol{k})\\0\end{pmatrix}. \tag{D1}$$

These are microscopic conservation laws for $\widetilde{\mathcal{N}}(\Gamma;\boldsymbol{k})$ and $\widetilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{k})$. For granular fluids, the energy density has a source $\tilde{w}(\Gamma;\boldsymbol{k})$ due to the inelasticity of the collisions. The forms of the fluxes of $\widetilde{\mathcal{N}}(\Gamma;\boldsymbol{k})$ and $\widetilde{\boldsymbol{\mathcal{G}}}(\boldsymbol{k})$ are obtained from

$$L\widetilde{\mathcal{N}}(\Gamma;\boldsymbol{k})=\frac{i}{m}\boldsymbol{k}\cdot\widetilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{k}),\quad L\widetilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{k})=i\boldsymbol{k}\cdot\tilde{\mathsf{h}}(\Gamma;\boldsymbol{k}). \tag{D2}$$

The expression for the tensor momentum flux $\tilde{\mathsf{h}}$ is

$$\begin{aligned}\tilde{\mathsf{h}}_{ij}(\Gamma;\boldsymbol{k})&=\sum_{r=1}^{N}mv_{r,i}v_{r,j}e^{i\boldsymbol{k}\cdot\boldsymbol{q}_r}\\&\quad+\frac{1}{2}\int_0^1dx\sum_{r=1}^{N}\sum_{s\neq r}^{N}q_{rs,i}F_j(\boldsymbol{q}_{rs},\boldsymbol{g}_{rs})e^{i\boldsymbol{k}\cdot(x\boldsymbol{q}_{rs}+\boldsymbol{q}_s)}.\end{aligned} \tag{D3}$$

This is the usual result for nonsingular forces $\boldsymbol{F}$, generalized here to include a nonconservative contribution as well. Some examples are discussed in Appendix B. In all of this appendix only nonsingular forces are considered. The corresponding results for hard spheres are given in the following companion paper.

The right sides of Eqs. (D2) are proportional to k, indicating that they are densities of conserved variables. For the energy density there is both a flux and a source,

$$L\mathcal{E}(\Gamma;\boldsymbol{k})=i\boldsymbol{k}\cdot\tilde{\boldsymbol{s}}(\Gamma;\boldsymbol{k})-\tilde{w}(\Gamma;\boldsymbol{k}). \tag{D4}$$

The energy flux is given by

$$\begin{aligned}\tilde{\boldsymbol{s}}(\Gamma;\boldsymbol{k})&=\sum_{r=1}^{N}\left(\frac{mv_r^2}{2}+\frac{1}{2}\sum_{s\neq r}^{N}V(q_{rs})\right)\boldsymbol{v}_re^{i\boldsymbol{k}\cdot\boldsymbol{q}_r}\\&\quad+\frac{1}{4}\int_0^1dx\sum_{r=1}^{N}\sum_{s\neq r}^{N}\boldsymbol{q}_{rs}(\boldsymbol{v}_r+\boldsymbol{v}_s)\cdot\boldsymbol{F}(\boldsymbol{q}_{rs},\boldsymbol{g}_{rs})e^{i\boldsymbol{k}\cdot(x\boldsymbol{q}_{rs}+\boldsymbol{q}_s)}\end{aligned} \tag{D5}$$

and the source term is

$$\widetilde{w}(\Gamma;\mathbf{k}) = -\frac{1}{2}\sum_{r=1}^{N}\sum_{s\neq r}^{N}\mathbf{g}_{rs}\cdot\mathbf{F}^{\rm nc}(\mathbf{q}_{rs},\mathbf{g}_{rs})e^{i\mathbf{k}\cdot\mathbf{q}_r}, \tag{D6}$$

where $\mathbf{F}^{\rm nc}(\mathbf{q}_{rs},\mathbf{g}_{rs})$ is the nonconservative part of the force. The functional forms for the fluxes $\widetilde{\mathbf{G}}(\Gamma;\mathbf{k})$, $\widetilde{s}(\Gamma;\mathbf{k})$, and $\widetilde{\mathsf{h}}(\Gamma;\mathbf{k})$ are the same as those for a normal fluid, except that the total force, including its nonconservative part, occurs. The source $\widetilde{w}(\Gamma;\mathbf{k})$ depends only on the nonconservative part of the force. For the special case of the force given in Eq. (B1), Eq. (D6) becomes

$$\widetilde{w}(\Gamma;\mathbf{k}) = \frac{1}{2}\sum_{r=1}^{N}\sum_{s\neq r}^{N}\Theta(\sigma-q_{rs})\gamma(\sigma-q_{rs})(\mathbf{g}_{rs}\cdot\hat{\mathbf{q}}_{rs})^2 e^{i\mathbf{k}\cdot\mathbf{q}_r}, \tag{D7}$$

which agrees with Eq. (B6) for $\mathbf{k}=0$.

The corresponding fluxes associated with the $\widetilde{a}_\alpha(\Gamma;n,T;\mathbf{k})$ follow from their definition, Eq. (64), in terms of the above densities,

$$L\widetilde{a}_\alpha(\Gamma;n,T;\mathbf{k}) = i\mathbf{k}\cdot\widetilde{\mathbf{f}}_\alpha(\Gamma;n,T;\mathbf{k}) - \frac{1}{e_{0,T}}\widetilde{w}(\Gamma;\mathbf{k})\delta_{\alpha2}, \tag{D8}$$

with

$$\widetilde{\mathbf{f}}_1(\Gamma;n,T;\mathbf{k}) = \frac{\widetilde{\boldsymbol{\mathcal{G}}}(\Gamma;\mathbf{k})}{m}, \tag{D9}$$

$$\widetilde{\mathbf{f}}_2(\Gamma;n,T;\mathbf{k}) = \frac{1}{e_{0,T}}\left(\widetilde{\mathbf{s}}(\Gamma;\mathbf{k}) - \frac{e_{0,n}}{m}\widetilde{\boldsymbol{\mathcal{G}}}(\Gamma;\mathbf{k})\right), \tag{D10}$$

$$\widetilde{\mathsf{f}}_{3,ij}(\Gamma;n,T;\mathbf{k}) = \frac{\widetilde{\mathsf{h}}_{ij}(\Gamma;\mathbf{k})}{nm}. \tag{D11}$$

The last equation above gives the tensor flux associated with the vector $\mathbf{a}_3=\boldsymbol{\mathcal{G}}/nm$.

Next, calculate the quantity $\widetilde{\ell}(\Gamma;n,T;\mathbf{k})$ appearing in Eq. (85). Use of Eq. (83) gives directly Eq. (85) with

$$\begin{aligned}
\widetilde{\ell}(\Gamma;n,T;\mathbf{k}) &= \frac{1}{e_{0,T}}\widetilde{w}(\Gamma;\mathbf{k}) + \zeta_0(n,T)T\frac{\partial}{\partial T}\widetilde{a}_2(\Gamma;n,T;\mathbf{k}) - \sum_\beta \mathcal{K}^{\rm hyd}_{2\beta}(n,T;0)\widetilde{a}_\beta(\Gamma;n,T;\mathbf{k})\\
&= \frac{1}{e_{0,T}}\widetilde{w}(\Gamma;\mathbf{k}) - \frac{\zeta_0(n,T)T}{e_{0,T}}\frac{\partial e_{0,T}}{\partial T}\widetilde{a}_2(\Gamma;n,T;\mathbf{k}) - \frac{\zeta_0(n,T)T}{e_{0,T}}\frac{\partial e_{0,n}}{\partial T}\widetilde{a}_1(\Gamma;n,T;\mathbf{k})\\
&\quad - \frac{\partial[\zeta_0(n,T)T]}{\partial n}\widetilde{a}_1(\Gamma;n,T;\mathbf{k}) - \frac{\partial[\zeta_0(n,T)T]}{\partial T}\widetilde{a}_2(\Gamma;n,T;\mathbf{k})\\
&= \frac{1}{e_{0,T}}\left[\widetilde{w}(\Gamma;\mathbf{k}) - \left(\frac{\partial[e_{0,T}\zeta_0(n,T)T]}{\partial n}\right)_T\widetilde{a}_1(\Gamma;n,T;\mathbf{k}) - \left(\frac{\partial[e_{0,T}\zeta_0(n,T)T]}{\partial T}\right)_n\widetilde{a}_2(\Gamma;n,T;\mathbf{k})\right].
\end{aligned} \tag{D12}$$

From the expression for the cooling rate in the HCS given in Eq. (48),

$$\begin{aligned}
e_{0,T}\zeta_0(n,T)T &= -V^{-1}\int d\Gamma\ \rho_h(\Gamma;n,T)LE(\Gamma)\\
&= V^{-1}\int d\Gamma\ \rho_h(\Gamma;n,T)\widetilde{w}(\Gamma;\mathbf{0})
\end{aligned} \tag{D13}$$

and, since $\widetilde{w}(\Gamma;\mathbf{0})$ is independent of n and T as seen from its definition in Eq. (D1),

$$\left(\frac{\partial(e_{0,T}\zeta_0T)}{\partial n}\right)_T = V^{-1}\int d\Gamma\ \Psi_1(\Gamma;n,T)\widetilde{w}(\Gamma;\mathbf{0}),$$

$$\left(\frac{\partial(e_{0,T}\zeta_0T)}{\partial T}\right)_n = V^{-1}\int d\Gamma\ \Psi_2(\Gamma;n,T)\widetilde{w}(\Gamma;\mathbf{0}). \tag{D14}$$

Substitution of the above relations into Eq. (D12), noting that the sum can be extended to include $\alpha=3$, since the new contribution vanishes by symmetry as a consequence of $\widetilde{w}$ being a scalar, gives Eq. (85) in the text.

2. Fluxes associated with $\widetilde{\psi}_\alpha(\Gamma;n,T;\mathbf{k})$

From Eq. (74), it is seen that the set of $\psi_\alpha(\Gamma;n,T;\mathbf{r})$ are densities associated with the invariants, i.e.,

$$\widetilde{\psi}_\alpha(\Gamma;n,T;\mathbf{0}) = \Psi_\alpha(\Gamma;n,T), \tag{D15}$$

and so

$$\sum_\beta \mathcal{U}_{\alpha\beta}(t,T)\widetilde{\psi}_\beta(\Gamma;n,T;\mathbf{0}) = \widetilde{\psi}_\alpha(\Gamma;n,T;\mathbf{0}) \tag{D16}$$

and

$$\partial_t\widetilde{\psi}_\alpha(\Gamma;n,T;\mathbf{0},t) = 0. \tag{D17}$$

Note that $\mathbf{U}$ has been set equal to zero and it has been suppressed in the notation. The time derivative of $\widetilde{\psi}_\alpha(\Gamma;n,T;\mathbf{k},t)$ must be of order k, so there exists a flux $\widetilde{\gamma}_\beta(\Gamma;n,T;\mathbf{k},t)$ such that

$$\partial_t \tilde{\psi}_\alpha(\Gamma;n,T;\boldsymbol{k},t) - i\boldsymbol{k}\cdot\tilde{\gamma}_\alpha(\Gamma;n,T;\boldsymbol{k},t) = 0. \tag{D18}$$

The generator of the dynamics $\mathcal{U}(t,T)$ is defined by Eq. (62), and taking the time derivative there, it is seen to obey the equation

$$\partial_t \mathcal{U}(t,T) = -[\overline{\mathcal{L}}_T - \mathcal{K}^{\text{hyd T}}(n,T;\boldsymbol{0})]\mathcal{U}(t,T), \tag{D19}$$

where $\mathcal{K}^{\text{hyd T}}$ is the transpose of $\mathcal{K}^{\text{hyd}}$. This equation can be formally integrated to write

$$\mathcal{U}(t,T) = \exp\{-t[\overline{\mathcal{L}}_T - \mathcal{K}^{\text{hyd T}}(n,T;\boldsymbol{0})]\}, \tag{D20}$$

which shows that Eq. (D19) is equivalent to

$$\partial_t \mathcal{U}(t,T) = -\mathcal{U}(t,T)[\overline{\mathcal{L}}_T - \mathcal{K}^{\text{hyd T}}(n,T;\boldsymbol{0})]. \tag{D21}$$

Use of this in Eq. (94) yields

$$\partial_t \tilde{\psi}(\Gamma;n,T;\boldsymbol{k},t) = -\mathcal{U}(t,T)[\overline{\mathcal{L}}_T - \mathcal{K}^{\text{hyd T}}(n,T;\boldsymbol{0})]\tilde{\psi}(\Gamma;n,T;\boldsymbol{k}). \tag{D22}$$

Comparison with Eq. (D18) leads to the identifications given in Eqs. (100) and (101).

APPENDIX E: DETAILS OF EULER ORDER PARAMETERS

The Euler order time-independent correlation function $\hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{D}}_{\alpha\beta}(n,T;\boldsymbol{0},0)$, whose expression is given in Eq. (132), is determined from direct evaluation. Consider first the case $\alpha=1$ for which $\boldsymbol{f}_1(\Gamma;n,T;\boldsymbol{0})=\tilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{0})/m$. Then

$$\begin{aligned}\hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{D}}_{1\beta}(n,T;\boldsymbol{0},0) &= (mV)^{-1}\int d\Gamma\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{0})\left[\frac{\partial\rho_h(\Gamma;n,T,\boldsymbol{U})}{\partial y_\beta}\right]_{\boldsymbol{U}=\boldsymbol{0}}\\ &= (mV)^{-1}\left[\frac{\partial}{\partial y_\beta}\int d\Gamma\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{0})\rho_h(\Gamma;n,T,\boldsymbol{U})\right]_{\boldsymbol{U}=\boldsymbol{0}}\\ &= V^{-1}\left[\frac{\partial}{\partial y_\beta}N\hat{\boldsymbol{k}}\cdot\boldsymbol{U}\right]_{\boldsymbol{U}=\boldsymbol{0}}\\ &= \delta_{\beta 3}n,\end{aligned} \tag{E1}$$

in agreement with Eq. (130). For $\alpha=2$, use Eq. (D10) to get

$$\begin{aligned}\hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{D}}_{2\beta}(n,T;\boldsymbol{0},0) &= (Ve_{0,T})^{-1}\int d\Gamma\hat{\boldsymbol{k}}\cdot\left(\tilde{\boldsymbol{s}}(\Gamma;\boldsymbol{0}) - \frac{e_{0,n}}{m}\tilde{\boldsymbol{\mathcal{G}}}(\Gamma;0)\right)\left[\frac{\partial\rho_h(\Gamma;n,t,\boldsymbol{U})}{\partial y_\beta}\right]_{\boldsymbol{U}=\boldsymbol{0}}\\ &= (Ve_{0,T})^{-1}\left[\frac{\partial}{\partial y_\beta}\int d\Gamma\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{s}}(\Gamma;\boldsymbol{0})\rho_h(\Gamma;n,T,\mathbf{U})\right]_{\boldsymbol{U}=\boldsymbol{0}} - e_{0,n}(mVe_{0,T})^{-1}\left[\frac{\partial}{\partial y_\beta}\int d\Gamma\hat{\boldsymbol{k}}\cdot\tilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{0})\rho_h(\Gamma;n,T,\boldsymbol{U})\right]_{\boldsymbol{U}=\boldsymbol{0}}.\end{aligned} \tag{E2}$$

The second term on the right-hand side is easily evaluated using the result in Eq. (E1). The ensemble average in the first term can be carried out by making the change of velocity variables $\boldsymbol{v}_r\rightarrow\boldsymbol{v}_r+\boldsymbol{U}$ and using the properties of the Galilean transformation,

$$\begin{aligned}\int d\Gamma\tilde{\boldsymbol{s}}(\Gamma;\boldsymbol{0})\rho_h(\Gamma;n,T,\boldsymbol{U}) &= \int d\Gamma[\tilde{\boldsymbol{s}}(\Gamma;\boldsymbol{0})]_{\{\boldsymbol{v}_r\rightarrow\boldsymbol{v}_r+\boldsymbol{U}\}}\rho_h(\Gamma;n,T,\boldsymbol{U}=\boldsymbol{0})\\ &= \int d\Gamma\left[\tilde{\boldsymbol{s}}(\Gamma;\boldsymbol{0}) + \left(\mathcal{E}(\Gamma;\boldsymbol{0}) + \frac{1}{2}mNU^2\right)\boldsymbol{U} + \tilde{\mathsf{h}}(\Gamma;\boldsymbol{0})\cdot\boldsymbol{U} + \frac{1}{2}\tilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{0})U^2 + \boldsymbol{U}\boldsymbol{U}\cdot\tilde{\boldsymbol{\mathcal{G}}}(\Gamma;\boldsymbol{0})\right]\rho_h(\Gamma;n,T)\\ &= e_0(n,T)V\boldsymbol{U} + \frac{1}{2}mNU^2\boldsymbol{U} + \int d\Gamma\tilde{\mathsf{h}}(\Gamma;\boldsymbol{0})\cdot\boldsymbol{U}\rho_h(\Gamma;n,T).\end{aligned} \tag{E3}$$

Using this result it is easily obtained that

$$\begin{aligned}\hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{D}}_{2\beta}(n,T;\boldsymbol{0},0) &= \frac{\delta_{\beta 3}}{e_{0,T}}\left(e_0 - e_{0,n}n + (Vd)^{-1}\int d\Gamma\ \text{tr}\ \mathsf{H}(\Gamma)\rho_h(\Gamma;n,T)\right).\end{aligned} \tag{E4}$$

Here $\mathsf{H}(\Gamma)\equiv\mathsf{h}(\Gamma,\boldsymbol{0})$, so tr $\mathsf{H}(\Gamma)=\Sigma_{i=1}^d\tilde{\mathsf{h}}_{ii}(\Gamma;\boldsymbol{0})$ is the volume-integrated momentum flux. For the case of the dissipative hard spheres discussed in Appendix B, it follows from Eq. (D3) that

$$\mathsf{H}_{ij}(\Gamma) = \sum_{r=1}^{N} mv_{r,i}v_{r,j} + \frac{1}{2}\sum_{r}^{N}\sum_{s\neq r}^{N} q_{rs,i}F_j(\boldsymbol{q}_{rs},\boldsymbol{g}_{rs}). \tag{E5}$$

For $\alpha=3$, use Eq. (D11) to get

$$
\begin{aligned}
\hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{D}}_{3\beta}(n,T;\boldsymbol{0},0)
&= (nmV)^{-1}\left[\frac{\partial}{\partial y_\beta}\int d\Gamma \mathsf{h}_{\|\|}(\Gamma;\boldsymbol{0})\rho_h(\Gamma;n,T,\boldsymbol{U})\right]_{\boldsymbol{U}=\boldsymbol{0}}\\
&= (nmV)^{-1}\left[\frac{\partial}{\partial y_\beta}\int d\Gamma[\mathsf{h}_{\|\,\|}(\Gamma;\boldsymbol{0}) + 2U_{\|}\mathcal{G}_{\|}(\Gamma;\boldsymbol{0})\right.\\
&\qquad \left. + mnU_{\|}^2]\rho_h(\Gamma;n,T,\boldsymbol{U}=\boldsymbol{0})\right]_{\boldsymbol{U}=\boldsymbol{0}},
\end{aligned}
\tag{E6}
$$

where again the change of velocity variables has been made. The subindex ‖ indicates the component in the direction of $\hat{\boldsymbol{k}}$. Therefore,

$$
\hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{D}}_{3\beta}(n,T;\boldsymbol{0},0) = (mn)^{-1}\left(\delta_{\beta 1}\frac{\partial}{\partial n} + \delta_{\beta 2}\frac{\partial}{\partial T}\right) \times (Vd)^{-1}\int d\Gamma\ \mathrm{tr}\ \mathsf{H}(\Gamma)\rho_h(\Gamma;n,T).
\tag{E7}
$$

These results are consistent with the form of the phenomenological matrix $\mathcal{K}_1^{\mathrm{hyd},(a)}$ in Eq. (131), if the pressure is identified as

$$
p(n,T) \equiv (Vd)^{-1}\int d\Gamma\ \mathrm{tr}\ \mathsf{H}(\Gamma)\rho_h(\Gamma;n,T).
\tag{E8}
$$

This is Eq. (133) in the main text.

The single transport coefficient at Euler order $\zeta^U(n,T)$ was identified in Eq. (117) as

$$
T\zeta^U(n,T) = -\lim \hat{\boldsymbol{k}}\cdot\overline{\boldsymbol{S}}_{23}^{(1)}(n,T;t),
\tag{E9}
$$

where the expression for $\boldsymbol{S}_{23}^{(1)}$ follows from Eq. (98),

$$
\overline{\boldsymbol{S}}_{23}^{(1)}(n,T;t) = V^{-1}\int d\Gamma\widetilde{\ell}^{(1)}(\Gamma;n,T)\Psi_3(\Gamma;n,T) - V^{-1}\int d\Gamma\widetilde{\ell}(\Gamma;n,T;\boldsymbol{0})\widetilde{\psi}_3^{(1)}(\Gamma;n,T;t).
\tag{E10}
$$

The first term on the right-hand side vanishes since

$$
\int d\Gamma\widetilde{\ell}^{(1)}(\Gamma;n,T)\Psi_3(\Gamma;n,T) = \left[\frac{\partial}{\partial U_{\|}}\int d\Gamma\widetilde{\ell}^{(1)}(\Gamma;n,T)\rho_h(\Gamma;n,T,\boldsymbol{U})\right]_{\boldsymbol{U}=\boldsymbol{0}}.
\tag{E11}
$$

The same change of variables $\boldsymbol{v}_r \to \boldsymbol{v}_r + \boldsymbol{U}$ as above shows that this vanishes due to spherical symmetry of $\rho_h(\Gamma;n,T,\boldsymbol{U}=\boldsymbol{0})$. The remaining contribution to $\zeta^U(n,T)$ as given by Eq. (E9) is made more explicit using the definition of $\widetilde{\ell}(\Gamma;n,T;\boldsymbol{0})$, Eq. (87), and also that of $\mathcal{U}_{3\alpha}(t,T)$, Eq. (62), to compute

$$
\widetilde{\psi}_3^{(1)}(\Gamma;n,T;t) = \mathcal{U}_{3\alpha}(t,T)\widetilde{\psi}_\alpha^{(1)}(\Gamma;n,T).
\tag{E12}
$$

This gives

$$
\begin{aligned}
\zeta^U(n,T)
&= +\lim(VT)^{-1}\int d\Gamma\ \widetilde{l}(\Gamma;n,T;\boldsymbol{0})\mathcal{U}_{3\alpha}(t,T)\hat{\boldsymbol{k}}\cdot\widetilde{\psi}_\alpha^{(1)}(\Gamma;n,T)\\
&= -\lim(VTe_{0,T})^{-1}\int d\Gamma[(1-P^\dagger)LE(\Gamma)]\\
&\quad \times\delta_{\alpha 3}e^{-\mathcal{L}_T t}\hat{\boldsymbol{k}}\cdot\widetilde{\psi}_\alpha^{(1)}(\Gamma;n,T)\\
&= \lim(VTe_{0,T}d)^{-1}\int d\Gamma\ W(\Gamma;n,T)e^{-\mathcal{L}_T t}\mathcal{M}_{\zeta^U}(\Gamma;n,T).
\end{aligned}
\tag{E13}
$$

The phase function $W(\Gamma;n,T)$ is defined by

$$
\begin{aligned}
W(\Gamma;n,T) &= -(1-P^\dagger)LE(\Gamma)\\
&= -LE(\Gamma) - N\left(\frac{\partial}{\partial n}(e_{0,T}T\zeta_0)\right)_{e_0} - E(\Gamma)\\
&\quad \times\left(\frac{\partial}{\partial e_0}(e_{0,T}T\zeta_0)\right)_n.
\end{aligned}
\tag{E14}
$$

The second equality follows from explicitly evaluating the action of the projection operator $P^\dagger$, defined in Eq. (88), on $LE(\Gamma)$. Finally, the phase function $\mathcal{M}_{\zeta^U}(\Gamma;n,T)$ is

$$
\begin{aligned}
\mathcal{M}_{\zeta^U}(\Gamma;n,T) &= d\hat{\boldsymbol{k}}\cdot\widetilde{\psi}_3^{(1)}(\Gamma;n,T)d\\
&= d\int d\boldsymbol{r}\ r_{\|}\left[\frac{\delta\rho_{\ell h}}{\delta U_{\|}(\boldsymbol{r})}\right]_{\{y_\beta\}=\{n,T,\boldsymbol{0}\}}\\
&= -\sum_{r=1}^{N}\boldsymbol{q}_r\cdot\frac{\partial\rho_h(\Gamma;n,T)}{\partial\boldsymbol{v}_r}.
\end{aligned}
\tag{E15}
$$

In the last transformation, the local equilibrium form for the velocity dependence has been taken into account,

$$
[\rho_{lh}(\Gamma|\{y_\beta\})]_{\{y_\beta\}=\{n,T,\boldsymbol{U}(\boldsymbol{r})\}} = \rho_h[\{\boldsymbol{q}_r,\boldsymbol{v}_r-\boldsymbol{U}(\boldsymbol{q}_r)\};n,T].
\tag{E16}
$$

APPENDIX F: NAVIER-STOKES ORDER TRANSPORT COEFFICIENTS

The Helfand forms for the transport coefficients are identified from (118) and (119). For the energy flux, these are the thermal conductivity λ and the new granular fluid coefficient μ,

$$
\lambda = e_{0,T}\lim\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\overline{\mathsf{D}}_{22}^{(1)}(n,T;t) - \sum_\alpha\overline{\boldsymbol{D}}_{2\alpha}(n,T;\boldsymbol{0},0)\overline{\boldsymbol{C}}_{\alpha 2}^{(1)}(n,T;t)\right),
\tag{F1}
$$

$$\mu = e_{0,T} \lim \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\overline{\mathsf{D}}_{21}^{(1)}(n,T;t) - \sum_{\alpha} \overline{\boldsymbol{D}}_{2\alpha}(n,T;\mathbf{0},0)\overline{\boldsymbol{C}}_{\alpha 1}^{(1)}(n,T;t)\right). \tag{F2}$$

The shear and bulk viscosities, η and κ, are identified as

$$\eta = mn \lim \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\overline{\mathsf{D}}_{44}^{(1)}(n,T;t) - \sum_{\alpha} \overline{\boldsymbol{D}}_{4\alpha}(n,T;\mathbf{0},0)\overline{\boldsymbol{C}}_{\alpha 4}^{(1)}(n,T;t)\right), \tag{F3}$$

$$\kappa + \frac{2(d-1)\eta}{d} = mn \lim \hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\overline{\mathsf{D}}_{33}^{(1)}(n,T;t) - \sum_{\alpha} \overline{\boldsymbol{D}}_{3\alpha}(n,T;\mathbf{0},0)\overline{\boldsymbol{C}}_{\alpha 3}^{(1)}(n,T;t)\right). \tag{F4}$$

Finally, the two Navier-Stokes transport coefficients associated with the cooling rate are

$$\zeta^{n} = T^{-1} \lim[\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\mathsf{S}_{21}^{(2)}(n,T;t) + T\zeta^{U}(n,T)\hat{\boldsymbol{k}} \cdot \boldsymbol{C}_{31}^{(1)}(n,T;t)], \tag{F5}$$

$$\zeta^{T} = T^{-1} \lim[\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\mathsf{S}_{22}^{(2)}(n,T;t) + T\zeta^{U}(n,T)\hat{\boldsymbol{k}} \cdot \boldsymbol{C}_{32}^{(1)}(n,T;t)]. \tag{F6}$$

The corresponding Green-Kubo forms are

$$\lambda = e_{0,T}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\mathsf{D}_{22}^{(1)}(n,T;\mathbf{0}) - \lim \int_0^t dt' \mathsf{G}_{22}(n,T;t')\right), \tag{F7}$$

$$\mu = e_{0,T}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\mathsf{D}_{21}^{(1)}(n,T;\mathbf{0}) - \lim \int_0^t dt' \mathsf{G}_{21}(n,T;t')\right), \tag{F8}$$

$$\eta = nm\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\mathsf{D}_{44}^{(1)}(n,T;\mathbf{0}) - \lim \int_0^t dt' \mathsf{G}_{44}(n,T;t')\right), \tag{F9}$$

$$\kappa + \frac{2(d-1)\eta}{d} = nm\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\mathsf{D}_{33}^{(1)}(n,T;\mathbf{0}) - \lim \int_0^t dt' \mathsf{G}_{33}(n,T;t')\right), \tag{F10}$$

$$\zeta^{n} = T^{-1}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\mathsf{S}_{21}^{(2)}(n,T;\mathbf{0}) - \lim \int_0^t dt'[\mathsf{N}_{21}^{(1)}(n,T;t') + T\zeta^{U}(n,T)\hat{\boldsymbol{k}}\boldsymbol{E}_{31}(n,T;\mathbf{0},t')]\right), \tag{F11}$$

$$\zeta^{T} = T^{-1}\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}:\left(\mathsf{S}_{22}^{(2)}(n,T;\mathbf{0}) - \lim \int_0^t dt'[\mathsf{N}_{22}^{(1)}(n,T;t') + T\zeta^{U}(n,T)\hat{\boldsymbol{k}}\boldsymbol{E}_{32}(n,T;\mathbf{0},t')]\right). \tag{F12}$$

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